

## 2.14 Scalar product

The formal definition of a scalar product we already stated in definition 14. It holds for vectors with real and complex coefficients as well as for functions interpreted as vectors (see section 3.8).

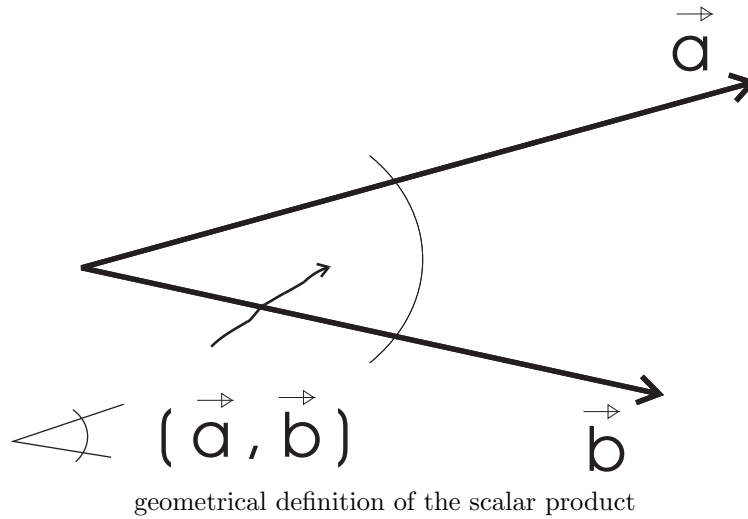
For an orthonormal set of base vectors  $\vec{e}_i$  and directly from the representation of a complex number in the Riemann-plane using the definition 6, we get

$$\langle \vec{a} | \vec{e}_i \rangle = a_i = |a| \cos \angle(\vec{a}, \vec{e}_i),$$

i.e. the scalar product between a vector  $\vec{a}$  and an orthonormal base vector  $\vec{e}_i$  calculates the projection of the vector  $\vec{a}$  in the direction of the base vector  $\vec{e}_i$ . This is the most important property of the scalar product. Directly from this statement and the linearity of the scalar product we get

$$\langle \vec{a} | \vec{a} \rangle = |\vec{a}|^2.$$

Example in 2D:  $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$  but also:  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \angle(\vec{a}, \vec{b})$



The scalar product can be interpreted as a special way of matrix multiplication:

$$\begin{aligned} \vec{a}^\top &\hat{=} 1 \times N = (a_1, \dots, a_N) \\ \vec{b} &\hat{=} 1 \times N = (b_1, \dots, b_N)^\top = \vec{b} \\ \vec{a} \cdot \vec{b} &= \vec{a}^\top \vec{b} = (a_1, \dots, a_N) \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} \hat{=} 1 \times 1 \quad . \end{aligned}$$

This interpretation is especially useful for the generalization of the scalar product with complex coefficients:

$$\vec{a} \cdot \vec{b} = \vec{a}^+ \cdot \vec{b} = \vec{a}^\top \vec{b} = \sum_{j=1}^N \bar{a}_j b_j$$

Example:

$$\begin{aligned} \vec{a} &= \begin{pmatrix} 1 \\ i \end{pmatrix} \\ \vec{b} &= \begin{pmatrix} i \\ i \end{pmatrix} \end{aligned} \quad \rightarrow \quad \langle a | b \rangle = \vec{a} \cdot \vec{b} = (1 - i) \begin{pmatrix} i \\ i \end{pmatrix} = +i + 1$$