2.12 Systems of Linear Equations

Example:

$$\begin{array}{rcrcrcrl} x_1 & + & 4x_2 & + & x_3 & = & 1 & (i) \\ 5x_1 & + & 4x_2 & - & x_3 & = & 2 & (ii) \quad x_1, x_2, x_3 = ? \\ 2x_1 & + & x_2 & + & x_3 & = & 0 & (iii) \end{array}$$

$$\begin{array}{lcll} (i)+(ii) & \Rightarrow & 6x_1 + 8x_2 = 3 & \mid \cdot 5 \\ & & & \quad - \\ (ii)+(iii) & \Rightarrow & \underline{7x_1 + 5x_2 = 2} & \mid \cdot 8 \\ & & -26x_1 = -1 & \end{array}$$

$$\begin{array}{ll} \to & x_1 = \frac{1}{26} \\ \to & x_2 = \frac{9}{26} \\ \to & x_3 = -\frac{11}{26} \end{array}$$

now in matrix notation:

$$\begin{pmatrix} 1 & 4 & 1 \\ 5 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 4 & 1 \\ 5 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix}^{-1}}_{\text{Problem, inverse of coefficient matrix } \tilde{A}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\det(\tilde{A}) = -26$$

$$\tilde{A}^{-1} = -\frac{1}{26} \begin{pmatrix} 5 & -7 & -3 \\ -3 & -1 & 7 \\ -8 & 6 & -16 \end{pmatrix}^{\top} = -\frac{1}{26} \begin{pmatrix} 5 & -3 & -8 \\ -7 & -1 & 6 \\ -3 & 7 & -16 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\frac{1}{26} \begin{pmatrix} 5 & -3 & -8 \\ -7 & -1 & +6 \\ -3 & +7 & -16 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = -\frac{1}{26} \begin{pmatrix} -1 \\ -9 \\ 11 \end{pmatrix} = \begin{pmatrix} \frac{1}{26} \\ \frac{9}{26} \\ \frac{-11}{26} \end{pmatrix}$$

In general: system of M linear equations with N variables $x_1, \ldots, x_N$.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1N}x_N & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2N}x_N & = & b_2 \\ \vdots & + & \vdots & + & \cdots & + & \vdots & = & \vdots \\ a_{M1}x_1 & + & a_{M2}x_2 & + & \cdots & + & a_{MN}x_N & = & b_M \end{array}$$

Matrix-Notation: $\tilde{A}\vec{x} = \vec{b}$

$$\tilde{A}\,(a_{jk}) \;\; M \times N, \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \; N \times 1, \quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_M \end{pmatrix} \; M \times 1$$

different cases:

(i) $M = N$ and $\det(\tilde{A}) \neq 0$ then: solution given by $\vec{x} = \tilde{A}^{-1}b$ (see above)

(ii) $M = N$ and $\det(\tilde{A}) = 0$
$\to$ in general no solution if $\vec{b} \neq \vec{0}$
$\to$ $\vec{x} = \vec{0}$ trivial solution if $\vec{b} = \vec{0}$ (homogeneous system) and further for any non-trivial solution $\vec{x}$ the vector $\alpha\vec{x}$ is another solution. If $\vec{x}^{(1)}, \ldots, \vec{x}^{(k)}$ are k linear independent solutions than $\vec{x} = k_1\vec{x}^{(1)} + \ldots + k_k\vec{x}^{(k)}$ with arbitrary $k_1, \ldots, k_k$ is another solution.

(iii) $M > N$: more equations than variables$\rightarrow$ in general no solution
solutions only if equations are trivially dependent on each other

$$\left.\begin{array}{l} x_1 + x_2 = 0 \\ \\ x_1 + 2x_2 = 1 \\ \\ 2x_1 + 2x_2 = 0 \end{array}\right\} \Rightarrow M = 3,\ N = 2 \text{ but effectively } M = N = 2$$

(iv) $M < N$: more variables than system of equations "over-determined "$\Rightarrow$ in general $N - M$ variables free or parameters
Example:

$$\begin{array}{rcll} x_1 + 3x_2 + 3x_3 + 4x_4 & = & 1 & (i) \\ & & & M = 2,\ N = 4 \\ -2x_1 + 3x_2 - 4x_3 + x_4 & = & 2 & (ii) \end{array}$$

$$(i) - (ii) \quad \Rightarrow \quad 3x_1 + 7x_3 - 3x_4 = -1 \quad \Rightarrow \quad x_1 = -\tfrac{7}{3}x_3 + x_4 - \tfrac{1}{3}$$

$$2(i) + (ii) \quad \Rightarrow \quad 9x_2 + 2x_3 + 9x_4 = 4 \quad \Rightarrow \quad x_2 = -\tfrac{2}{9}x_3 - x_4 + \tfrac{4}{9}$$

x_3, x_4 are free parameters.

Important Case: $M = N$
Cramer-Rule:

$$\begin{array}{c} a_{11}x_1 + \ldots + a_{1N}x_N = b_1 \\ \cdots \\ a_{N1}x_1 + \ldots + a_{NN}x_N = b_N \end{array}$$

If $\det(\tilde{A}) \neq 0$ then the solution is given by

$$x_1 = \frac{\det(\tilde{A}_1)}{\det(\tilde{A})}, \quad x_2 = \frac{\det(\tilde{A}_2)}{\det(\tilde{A})}, \ldots, x_N = \frac{\det(\tilde{A}_N)}{\det(\tilde{A})}$$

where

$$\tilde{A}_j = \begin{pmatrix} a_{11} & a_{12} & b_1 & \cdots & a_{1N} \\ a_{21} & a_{22} & b_2 & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & b_N & \cdots & a_{NN} \end{pmatrix} \text{ thus not very efficient for larger systems!!}$$
$$\uparrow$$

i.e. in the j-th column the vector $\begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}$ is placed instead of $\begin{pmatrix} a_{jj} \\ \vdots \\ a_{Nj} \end{pmatrix}$

Example as before:

$$\begin{array}{rcl} x_1 + 4x_2 + x_3 & = & 1 \\ 5x_1 + 4x_2 - x_3 & = & 2 \\ 2x_1 + x_2 + x_3 & = & 0 \end{array}$$

$$\det(\tilde{A}) = -26 \neq 0 \qquad \det(\tilde{A}_1) = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 4 & -1 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

$$\det(\tilde{A}_2) = \begin{vmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 2 & 0 & 1 \end{vmatrix} = -9 \qquad \det(\tilde{A}_2) = \begin{vmatrix} 1 & 4 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 11$$

$$\Rightarrow \quad x_1 = \frac{1}{26}, \quad x_2 = \frac{9}{26}, \quad x_3 = -\frac{11}{26}$$

There are efficient algorithms for large systems with numerically and analytically (Gauß Algorithm etc.) They all rely on the transformations of the matrix $\tilde{A}$ with respect to $\vec{b}$.