2.11 Classification of N x N Matrices:

$$\tilde{A} = \begin{pmatrix} a_{11} \cdots a_{1N} \\ \vdots \\ a_{N1} \cdots a_{NN} \end{pmatrix}$$

(i) $\det(\tilde{A}) \neq 0$ $\Rightarrow$ regular matrix

(ii) $\tilde{A} = \tilde{A}^\top$ $\Rightarrow$ symmetric matrix $a_{jk} = a_{kj}$ example $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

(iii) $\tilde{A} = -\tilde{A}^\top$ $\Rightarrow$ anti-symmetric matrix $a_{jk} = -a_{kj}$, in particular $a_{jj} = 0 \Rightarrow \text{tr}(\tilde{A}) = 0$

(iv) $\tilde{A} = \tilde{A}^+$ $\Rightarrow$ self-adjoined matrix or Hermite-matrix , this means:

$\tilde{A} = \overline{(\tilde{A}^\top)}$, $a_{jk} = \overline{a_{kj}}$

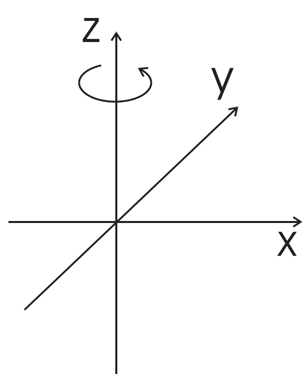
example: $\tilde{A} = \begin{pmatrix} 1 & 2 & -i \\ 2 & 2 & 1-i \\ i & 1+i & 3 \end{pmatrix}$,

if $\tilde{A}$ is real then (ii) and (iv) are equivalent. $\rightarrow$ diagonal elements of a Hermite-matrix are real, because of $a_{jj} = \overline{a_{jj}}$. The determinant is also real, i.e. $\det(\tilde{A}) \in \mathbb{R}$ if $\tilde{A}$ is a Hermite matrix.

(v) $\tilde{A} = -\tilde{A}^+$ $\Rightarrow$ anti-Hermite matrix ($\rightarrow$ diagonal elements vanish) in particular $a_{jj} = 0 \Rightarrow \text{tr}(\tilde{A}) = 0$

(vi) $\tilde{A}^\top = \tilde{A}^{-1}$ $\Rightarrow$ $\tilde{A}$ is called orthogonal (real case) $\tilde{A}\tilde{A}^\top = \tilde{A}^\top\tilde{A} = \tilde{I}$ if $\tilde{A}$ is complex than $\tilde{A}^+ = \tilde{A}^{-1}$ means that A is "unitary". Properties of orthogonal matrices: $\det(\tilde{A}) = \pm 1$ (follows from $\det(\tilde{A}) = \det(\tilde{A}^\top)$ and $\det(\tilde{A}\tilde{A}^\top) = \det(\tilde{A})\det(\tilde{A}^\top)$) if $\tilde{A}$ and $\tilde{B}$ orthogonal, then $\tilde{A} \cdot \tilde{B} = \tilde{C}$ is also orthogonal.

Example: rotation around z-axis



$$\begin{aligned} \tilde{A} &= \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \rightarrow \tilde{A}^\top &= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \Rightarrow \tilde{A}\tilde{A}^\top &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{since} \quad \cos^2\phi + \sin^2\phi = 1 \end{aligned}$$

Test:

$$\det(\tilde{A}) = 1 \begin{vmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{vmatrix} = +1 \text{ o.k.}$$

(vii) diagonal Matrix:

$$\tilde{A} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{NN} \end{pmatrix}$$

$\det(\tilde{A}) = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{NN}$ for diagonal matrix $\tilde{A}$