2.9 Square matrices and determinants

Now $N \times N$ matrices $\tilde{A} = (a_{jk})\ j,k = 1,\ldots,N$

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$$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix}$$

$$\det\left(\tilde{A}\right) = \begin{vmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{vmatrix} = \sum_{P(N)} (-1)^{j(P)}\, a_{1,j_1} a_{2,j_2} \cdots a_{N,j_N}$$

where P(N) are all permutations of the numbers $1,\ldots,N$
and j(P) is the number of changes between $(1,\ldots,N)$ *and* $(j_1,\ldots,j_N)$

$\Rightarrow$ *definition not practical for a calculation of* $\det(\tilde{A})$. *Therefore, calculation via successive expansion in sub-determinants (Laplace rule):* $N = 1:\ \det(a) = a \quad a \in \mathbb{R}$.

As we will see the determinant is the (only) totally antisymmetric multilinear operation acting on the components of a matrix. Corresponding to the Kronecker-symbol sometimes a notation using the totally antisymmetric function $\epsilon_{i,j,\ldots,k}$ (Levi-Civita symbol) is helpful for the formal calculation of a determinant:

$$\det\left(\tilde{A}\right) = \sum_{P(N)} (-1)^{j(P)}\, a_{1,j_1} a_{2,j_2} \cdots a_{N,j_N} = \sum_{j_1,j_2,\ldots,j_N=1}^{N} \epsilon_{j_1,j_2,\ldots,j_N} a_{1,j_1} a_{2,j_2} \cdots a_{N,j_N}$$

$\epsilon_{j_1,j_2,\ldots,j_N}$ is zero if any of the indices are equal, it is $1 = \epsilon_{1,2,\ldots,N}$, and changes it's sign for each change in the order of indices. (Hint: this are exactly the properties of the quantum numbers of Fermions according to the Pauli principle, the determinant or $\epsilon_{i,j,\ldots,k}$ are therefor often used to calculate many particle states in quantum mechanics).
The geometrical interpretation in 3D of a determinant will be given in section 2.15 and a more general interpretation in section 2.16.

Calculation by Laplace rule:

$$\det\left(\tilde{A}\right) = \sum_{j=1}^{N} a_{jk} A_{jk} = \sum_{j=1}^{N} a_{kj} A_{kj},\ \text{for } N > 1$$

development via the column/line, adaptive expansion by column or row where: $k \in (1,\ldots,N)$ arbitrary and cofactor of a_{jk} in $\tilde{A}$

$$A_{jk} = (-1)^{j+k} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,k-1} & a_{1,k+1} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2N} \\ \vdots & & & & & & \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,k-1} & a_{j-1,k+1} & \cdots & a_{j-1,N} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,k-1} & a_{j+1,k+1} & \cdots & a_{j+1,N} \\ \vdots & & & & & & \\ a_{N1} & a_{N2} & \cdots & a_{N,k-1} & a_{N,k+1} & \cdots & a_{NN} \end{vmatrix} \hat{=} \text{ determinants of } \tilde{A} \text{ where j}^{th} \text{ line and k}^{th} \text{column are erased}$$

Examples:

(i)

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \quad \text{development via } 1^{\text{st}} \text{ column}$$

(ii)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{12}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

$\Rightarrow$ calculation of larger determinants still difficult!

<u>Calculation rules for determinants:</u>

(a) Determinants are antisymmetric for changing the order of rows or columns,
i.e. $\begin{vmatrix} \vec{a} & \vec{b} & \cdots \end{vmatrix} = -\begin{vmatrix} \vec{b} & \vec{a} & \cdots \end{vmatrix}$

(b) Determinants vanish if two vectors are identical,
i.e. $\begin{vmatrix} \vec{a} & \vec{a} & \cdots \end{vmatrix} = -\begin{vmatrix} \vec{a} & \vec{a} & \cdots \end{vmatrix} = 0$

(c) Determinants are linear,
i.e. $\begin{vmatrix} \vec{a}+\vec{b} & \cdots \end{vmatrix} = \begin{vmatrix} \vec{a} & \cdots \end{vmatrix} + \begin{vmatrix} \vec{b} & \cdots \end{vmatrix}$

(d) Determinants are linear,
i.e. $\begin{vmatrix} \alpha\vec{a} & \cdots \end{vmatrix} = \alpha\begin{vmatrix} \vec{a} & \cdots \end{vmatrix}$

(e) Adding linear combination of other rows/columns does not change Determinants,
i.e. $\begin{vmatrix} \left(\vec{a}+\beta\vec{b}\right) & \vec{b} & \cdots \end{vmatrix} = \begin{vmatrix} \vec{a} & \vec{b} & \cdots \end{vmatrix} + \beta\begin{vmatrix} \vec{b} & \vec{b} & \cdots \end{vmatrix} = \begin{vmatrix} \vec{a} & \vec{b} & \cdots \end{vmatrix}$

(f) Subtracting projections of (row/column)-vectors does therefor not change the determinant, so the determinant calculates the product of the length of a set of orthogonal vectors, i.e the volume spanned up by the set of vectors. If the volume is not zero the set of vectors is linearly independent.