2.7 Matrices

Definition 19

$$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix} = (a_{jk})_{\substack{j=1,\dots,M \\ k=1,\dots,N}} \qquad \textit{with } a_{jk} \in \mathbb{R} \textit{ or } \mathbb{C}$$

$\tilde{A}$ is called a $M \times N$ matrix, M rows or lines, N columns. i.e. vectors are matrices!

The fact that a matrix according to the definition 13 acts as a linear transformation between vectors will be needed later.

Definition 20

- *Sum $\oplus$ of two $M \times N$ matrices $\tilde{A}$ and $\tilde{B}$*
  $\tilde{A} + \tilde{B} = \tilde{C}: \quad a_{jk} + b_{jk} = c_{jk}$
  zero element: $\tilde{A} = 0 \Rightarrow a_{jk} = 0$

$$\tilde{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \tilde{0}, \textit{ of course: } \tilde{A} + \tilde{B} = \tilde{B} + \tilde{A}$$

- *Scalar multiplication $\otimes$:* $\alpha \in \mathbb{R}$ *or* $\mathbb{C}$

$$\alpha\tilde{A} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1N} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2N} \\ \vdots & & \ddots & \vdots \\ \alpha a_{M1} & \alpha a_{M2} & \cdots & \alpha a_{MN} \end{pmatrix} = (\alpha a_{jk})$$

  Trivial: $\tilde{A} = \tilde{B}$ means: $a_{jk} = b_{jk}$ for all $j = 1, \dots, M$ and $k = 1, \dots, N$.
  If a_{jk} are complex numbers $a_{jk} = x_{jk} + iy_{jk}$ then simply

$$\begin{aligned} \tilde{A} &= \tilde{X} + i\tilde{Y} \quad \textit{where } \tilde{X} = (x_{jk}), \tilde{Y} = (y_{jk}) \\ \overline{\tilde{A}} &= \tilde{X} - i\tilde{Y} \quad \textit{complex conjugated matrix with respect to } \tilde{A} \end{aligned}$$

$M \cdot N$ base matrices $\begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ & \cdots & & \ddots \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & & \cdots & \\ & \cdots & & \cdots \end{pmatrix}$, etc.

(not important here, but:) $\Rightarrow$ Matrices form an $M \cdot N$ dimensional vector space.

Definition 21 *transposed matrix*

$$\tilde{A} = \overbrace{(a_{jk})}^{M\times N \textit{ matrix}}{}_{\substack{j=1,\dots,M \\ k=1,\dots,N}} \Rightarrow \tilde{A}^{\top} = \overbrace{(a_{kj})}^{N\times M \textit{ matrix}}{}_{\substack{k=1,\dots,N \\ j=1,\dots,M}}$$

<u>adjoined</u> matrix:

$a_{jk} \in \mathbb{C}\ \tilde{A} = (a_{jk}) \Rightarrow \tilde{A}^{+} = \left(\overline{\tilde{A}}\right)^{\top} = (\bar{a}_{kj}) = \left(\overline{\tilde{A}^{\top}}\right)$
the^{-} "bar" stands for the complex conjugated:

$$z = a + ib = re^{i\varphi}, \qquad \bar{z} = a - ib = re^{-i\varphi}$$

<u>Examples:</u>

$$\tilde{A} = \begin{pmatrix} 1 & 2 & i \\ 3 & 1+i & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \tilde{A}^{\top} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1+i & 0 \\ i & 0 & 2 \end{pmatrix} \quad \tilde{A}^{+} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1-i & 0 \\ -i & 0 & 2 \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow \tilde{A}^\top = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \left(\tilde{A} + \tilde{B}\right)^\top = \tilde{A}^\top + \tilde{B}^\top, \left(\tilde{A}^\top\right)^\top = \tilde{A}$$

most important: quadratic matrices, i.e. $M = N$

$$\tilde{A} = (a_{jk})_{j,\,k\,=\,1,\,...\,,\,N} = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix}}_{\substack{\text{main diagonal elements} \\ a_{jk} \text{ with } j=k}}$$

Definition 22 *trace of a quadratic matrix:* $tr\left(\tilde{A}\right) = \sum_{j=1}^{N} a_{jj} = a_{11} + a_{22} + \ldots + a_{NN}$
trivial: $tr\left(\tilde{A}\right) = tr\left(\tilde{A}^\top\right)$