Collisions

Science

General Consideration and Fully Elastic Collisions

Here I look at collisions of two bodies in some gruesome detail. It's just basic technical mechanics but that is gruesome enough for most. Much of what follows comes straight out of a text book [1)] for undergraduate engineering students.
First let's see what kind of collisions we will look at in increasing order of complexity:

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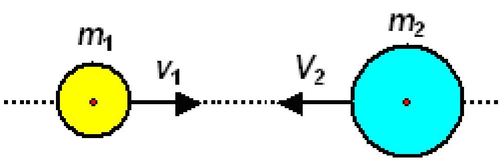
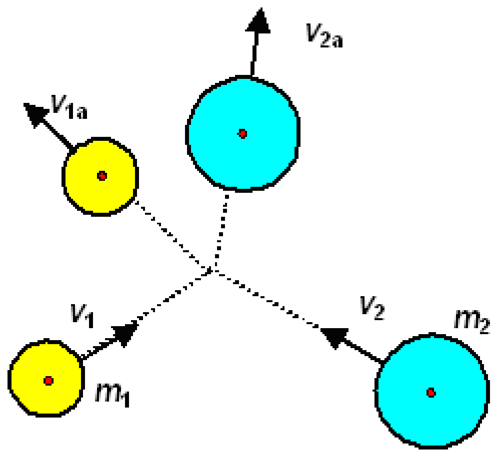
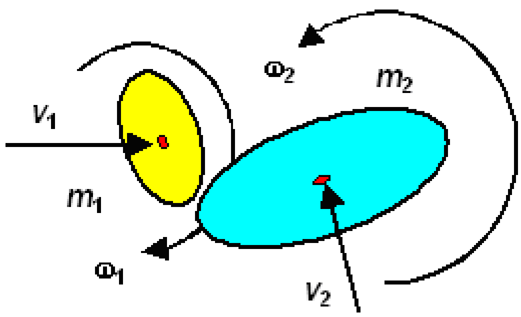
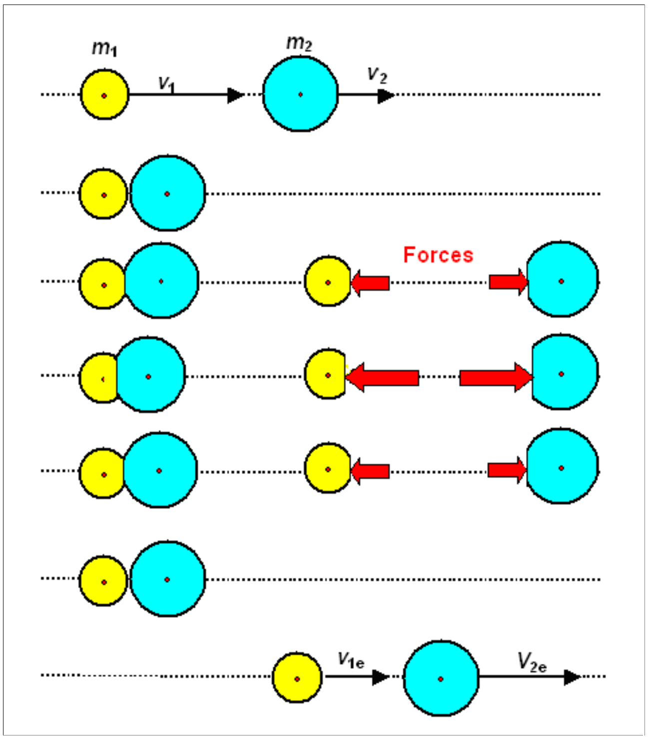
	1-dim. straight collision spheres / mass points
	2-dim. collision spheres / mass points
	2-dim general collision with real and rotating bodies
Kinds of collisions	

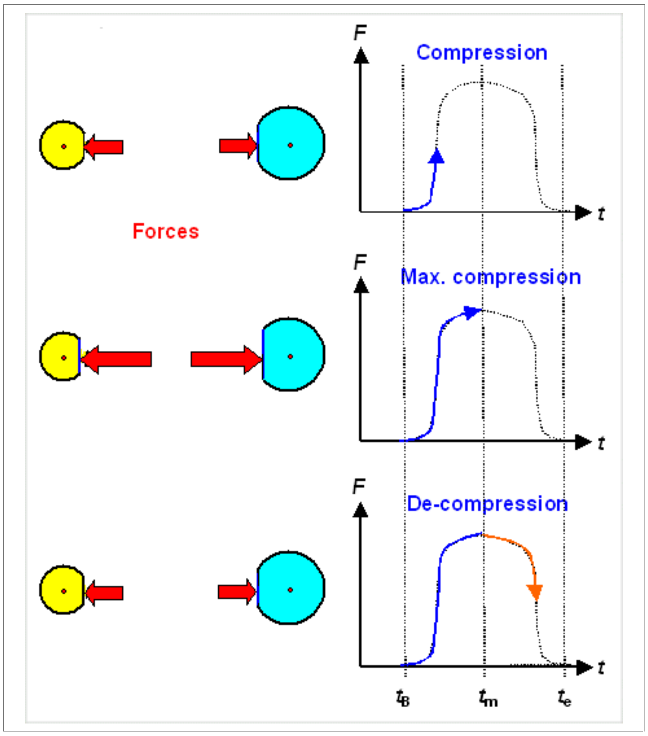
- To make things a bit more interesting we consider elastic or inelastic collision and anything in between.
  Be happy we don't look at three-dimensional collision and only at objects with smooth surfaces (no friction when touching) and not at rough surfaces with some friction.

For starters we look at the simple first case. To add some variety, I change the situation a little bit. The yellow sphere is now chasing the blue one. Since it is obviously faster, it will eventually hit the blue one from behind. The moment just before impact is shown on the second tier.



Before, during, and after impact

- Upon impact *forces* occur at $t=t_{begin}=t_b$ that will accelerate the blue sphere, decelerate the yellow one, and deform elastically both of them (third tier). These forces grow to some maximun coupled to maximum deformation (fourth tier) at $t=t_{max}=t_m$. That is an interesting point in time because the two spheres are now as close together as they ever will get and move at *exactly the same speed* .
  Think about it. There must be such at point in time and it can *only* be at maximum deformation. We call that "midpoint" speed v_{mid}
  Next the de-compression or restitution stage starts. The elastic deformation is *fully reversed*. The spheres push each other apart with forces more or less the mirror image of what made the defamations during the compression phase. These forces will act on the masses of the spheres, changing their velocities from the common one at the "midpoint" to the final one v_{1e} and v_{2e} encountered as soon as they separate again at $t=t_{end}=t_e$.
  This is shown with some (schematic) force over time diagram below:



Forces and deformations during the (very short) impact phase

From this consideration it is now easy to calculate what will happen for the perfectly elastic collision and - with one more simple thought - for the perfectly inelastic one, too. From that we also get the results for anything in between perfectly elastic or perfectly inelastic.
Firs, we notice that the integral over $F(t)dt$ always gives the *change in momentum* of whatever mass it acts on for the time span of the integration. Integrating from t_b to t_m thus gives the *total* **impact force** acting through the compression phase; we call that *impact force* F_{com}

- **Important!!**
  An ***impact force*** is *not* a force! It has the dimension **Ns** (force times time) and the symbol F still used by all and sundry then carries some "accent" like a little something on top. That cannot easily be produced with simple HTML, so I won't do it.
  In order to keep things easy, I just keep using the symbol F in what follows but always with some subscript like F_{com}

Similarly, integrating from t_m to t_e gives F_{decom} the *impact force* during de-compression.
It si clear that for fully elastic bodies, and therefore fully elastic collisions, the two impact forces must be identical Writing this down, together with the momentum changes during compression and decompression, yields

$$F_{com} = F_{decom}$$

$F_{com} = -m_1(v_{mid} - v_1)$ $F_{com} = +m_2(v_{mid} - v_2)$	**Compression phase**
$F_{decom} = -m_1(v_{1e} - v_{mid})$ $F_{decom} = +m_2(v_{2e} - v_{mid})$	**De-compression phase**

- Done. Five equations for the five unknowns v_{1e}. v_{2e}, v_{mid} , F_{com} and F_{decom}. Nothing to it. We obtain for the velocities (we don't care for the forces):

$$v_{1e} = \frac{2m_2 \cdot v_2 + (m_1 - m_2) \cdot v_1}{m_1 + m_2}$$

$$v_{2e} = \frac{2m_1 \cdot v_1 + (m_2 - m_1) \cdot v_2}{m_1 + m_2}$$

$$v_{mid} = \frac{m_1 \cdot v_1 + m_2 \cdot v_2}{m_1 + m_2}$$

- Rather simple. For equal masses the speeds are interchanged. Mass 1 moves with the initial speed of mass 2 and vice verse. You also see that for the mass m_2 at rest (i.e. v_2=0) and m_1 representing your sword that is hitting mass 2, we can look at two extreme cases:

  1. $m_2 << m_1$. We get $v_{1e}=v_1$ and $v_{2e}=2v_1$. You are hitting a fly and your don't even feel it, your sword just keeps moving. The fly is just sped up.
  2. $m_2 >> m_2$. We get $v_{2e}=0$ and $v_{1e}=-v_1$. You are hitting a large rock. Your sword will just bounce off, reversing its original speed. The rock isn't doing anything.

We have (check it!) momentum preservation (i.e. $m_1v_1 + m_2v_2 = m_1v_{1e} + m_2v_{2e}$) and *kinetic* energy preservation (since ther are no other energies involved), i.e. $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_{1e}^2 + \frac{1}{2}m_2v_{2e}^2$. That means that no damage of any kind occurred, the two objects after the impact are the same as before. Think of tennis rackets hitting tennis balls, golf clubs.... etc.. No damage to the bodies, only changes in velocities.

- I'm not going to spend more time on this but move on to the more interesting case of *fully inelastic* and "*in-between*" collisions.

Inelastic and "Mixed" Collisions

This topic is actually quite easy to deal with. All that happens is that there is no force F_{decom} for the purely inelastic case. What happens in this case looks like this:

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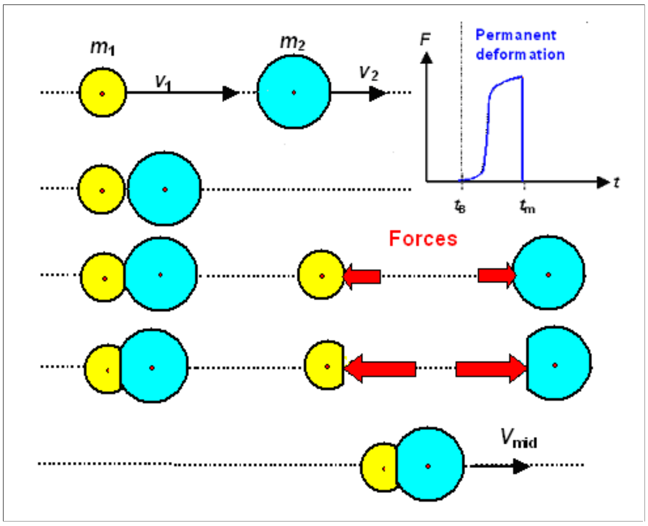


Fully inelastic collision

- Upon impact the two bodies deform as before but the deformation now is *purely plastic*, i.e. permanent. At the moment of maximum deformation, the two bodies move with the same speed v_{mid} and that's it! The faster yellow sphere was decelerated to v_{mid}, the slower one was accelerated to v_{mid}. Since there is no reversal of the deformation, there are no more forces after v_{mid} is reached (see inset in the picture above) and the two bodies now move in close contact with v_{mid}.
  That means we can use all the equations from above if we simply omit the force F_{decom} , i.e. $F_{decom}=0$

What about the *mixed* case? Neither fully elastic nor fully inelastic? That would be the proper way to look at things if you hit a soft (rather inelastic) target with your rather elastic blade.

- Well, in this case there will be a decompression phase with some force acting on both bodies. This force is, however, smaller than the force encountered during the compression phase. This changes our force equation like this

$$F_{decom} = e \cdot F_{com}$$

e=coefficient of restitution;
between 1 (fully elastic) and 0 (fully inelastic)

- So we are still left with 5 equations for 5 unknowns, we just have one more input parameter beside the two masses and the two initial velocities: the coefficient of restitution **e**.

What we get for the velocities after the impact is

$$v_{1e} = \frac{m_1 \cdot v_1 + m_2 \cdot v_2 - e \cdot (m_1 - m_2) \cdot v_1}{m_1 + m_2}$$

$$v_{2e} = \frac{m_1 \cdot v_1 + m_2 \cdot v_2 + e \cdot (m_1 - m_2) \cdot v_1}{m_1 + m_2}$$

- This allows to obtain a neat little relation between the velocity difference before and after a collision. We have

$$v_{2e} - v_{1e} = e \cdot (v_1 - v_2)$$

- For the purely inelastic collision (**e=0**) we get $v_{2e} = v_{1e}$. The two bodies now move with the same velocity, jsut as we already deduced abive.

You can check if the preservation of momentum is still observed, i.e. if $m_1 v_1 + m_2 v_2 = m_1 v_{1e} + m_2 v_{2e}$, and you will find that it is.
However, the total *kinetic* energy after the collision is no longer the same as before the collision if $e < 1$ obtains. This must be the case since some of the initial purely kinetic energy has beed transferred to the bodies and is no stored as "deformation" energy in the damaged bodies
This gives us a handle on calculating the **damage** done to the system. We define that as the energy E_{dam} that results from the difference of the (purely kinetic) energy $E_{kin;\,b}$ before the impact and the kinetic energy $E_{kin;\,e}$ after the impact, i.e.

$$E_{dam} = E_{kin;\,b} - E_{kin;\,e}$$
$$= ½(m_1 v_1^2 + m_2 v_2^2) - ½(m_1 v_{1e}^2 + m_2 v_{2e}^2)$$

$$E_{dam} = \frac{1 - e^2}{2} \cdot \frac{m_1 \cdot m_2}{m_1 + m_2} \cdot (v_1 - v_2)^2$$

- We can use this to define the **damage efficiency** η of a collision. It is the relation between the kinetic energy of the bodies before the collision and the Energy "used up" to produced the damage. We have

$$\eta = \frac{E_{kin;\,b}}{E_{dam}}$$

$$= 1 - e^2 \cdot \frac{m_2}{m_1 + m_2}$$

Bingo! What the equation tells you is simple. For an efficient hit with your sword against some target with the mass m_2 you should have:

- An inelastic collision, i.e. a soft inelastic target with a small **e** value
- A target mass m_2 that is much larger than the *effective* mass m_1 of your blade.

Let's look at a few situations where the efficiency equation can be applied:

1 You can't damage very small masses with your sword! The efficiency is essentially zero. In other words: Don't go after that mosquito with your katana. You might be able to move it around quite a bit but you cannot damage it. This joke [2] is thus quite funny but physically unsound. Wait until the mosquito sits down on a wall (moving up the target mass to near infinity; mosquito + wall + earth).

2 Forging should be done on an heavy anvil. It is the total weigh of the anvil and the material to be forged that counts. You want to deform your iron, i.e. damage it. Use a not too heavy hammer and hit with speed.

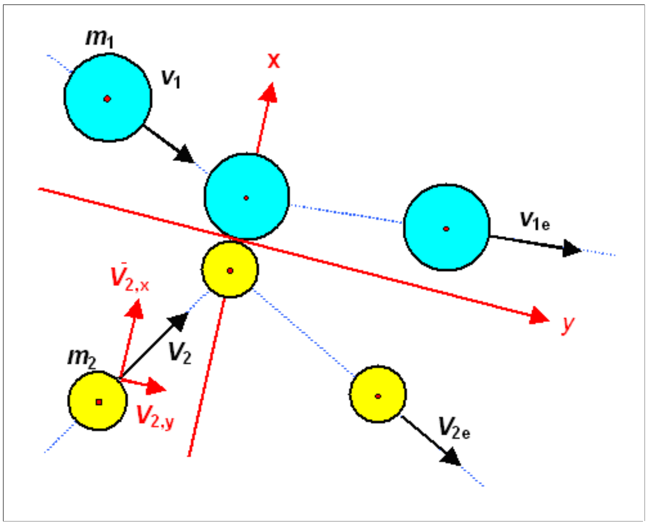
3 Drive that nail into the wood with a relatively heavy hammer. You don't want to damage the nail, you just want to give it speed so it can damage the wood.

4 Guns work. The projectile is very light and will thus inflict damage on anything still lightweight but much heavier than the projectile. It is extremely fast and thus contains a lot of energy despite its tiny mass, and that leads to severe damage.

5 Light swords do *not* work. They will transfer their energy very efficiently to everything substantially heavier, but there is not much energy to transfer. Modern guns have muzzle speeds of more than 1.000 m/s and that gives even a light weight bullet plenty of energy. You can't get close to that speed with your sword, and remember that speed counts quadratically.

Two-Dimensional Collisions

We willl just have a very a quick look at two-dimensional collisions between spheres (ensuring that we have a "central" collision). They meet at a point and we can always find a coordinate system as shown below. If the spheres have a smooth surface, meaning there is no friction, we are done.



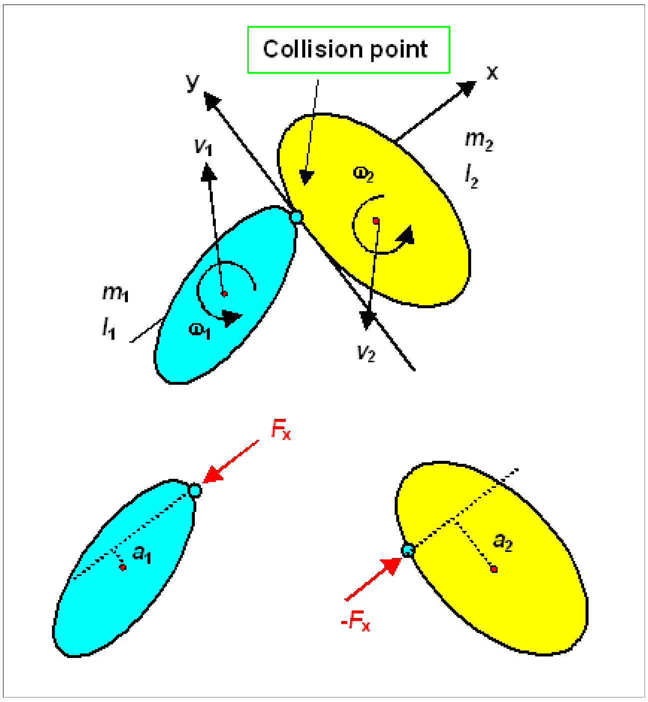
Two dimensional central collision
Choosing the coordinate system as shown is advantageous

- We can always find a coordinate system as shown. We then look at the ***x*** and ***y*** components of the velocities (as shown for **v2** separately and realize that we already have the solutions for the ***x*** components, and that nothing happens to the ***y*** components as long as there is no friction, i.e. we have spheres with "smooth" surfaces.

So just take the old solutions and add the **y**components of the initial velocities to the final ones. No more need to be said.

Arbitrary collision

Now we look at two arbitrary bodies **1** and **2**, with mass $m_{1,2}$ and moment of inertia $I_{1,2}$ (for the center of mass) with initial linear and rotational velocities like $\mathbf{v}_{1,2}$ and $\omega_{1,2}$. For simplicities sakes we assume smooth surfaces, i.e. no friction.
Once more we pick a smart coordinate system with the x-axis at right angles to the (self-explaining) collision "plane". The situation then looks like this:



Close to an arbitrary collision
F_x denotes the impact force

- The impact force F_x follows in principle from the integration over the compressing and de-copressing real forces $\mathbf{F_{com}}$ and $\mathbf{F_{decom}}$, related by $\mathbf{F_{decom}} = e \cdot \mathbf{F_{com}}$ as detailed above.

All we do is to write down the proper equations for the preservation of the (linear) momentum *and* of the angular momentum. To make things easier, we use impact forces right away and use ***e***, the coefficient of restitution from the beginning. We obtain:

$$
\begin{aligned}
m_{1x}(v_{1xe} - v_{1x}) &= -F_x \\
m_{1y}(v_{1ye} - v_{1y}) &= 0 \\
I_1(\omega_{1e} - \omega) &= +a_1 \cdot F_x \\
\\
m_{2x}(v_{2xe} - v_{2x}) &= +F_x \\
m_{2y}(v_{2ye} - v_{2y}) &= 0 \\
I_2(\omega_2 - \omega_{2e}) &= -a_1 \cdot F_x
\end{aligned}
$$

- The linear and angular velocities used relate to the center of mass (COM) of the two bodies. To calculate what happens upon impact, we also need the velocities $\mathbf{v_{1,2\,P}}$ at the impact point before and after the impact. They result form the linear COM velocity plus the velocity resulting at the impact point from the rotational velocity. This is easy, we get

$$
\begin{aligned}
v_{1xP} &= v_{1x} - a_1 \cdot \omega_1 & \quad v_{2xP} &= v_{2x} - a_2 \cdot \omega_2 \\
v_{1xeP} &= v_{1xe} - a_1 \cdot \omega_{1e} & \quad v_{2xeP} &= v_{2xe} - a_2 \cdot \omega_{2e}
\end{aligned}
$$

Done (almost) We have 11 equations for 11 unknowns, so get to work. We get, for example, for the total impact force F_x

$$F_x = (1+e)\frac{v_{1x} - a_1\omega_1 - (v_{2x} - a_2\omega_2)}{\frac{1}{m_1} + \frac{1}{m_2} + \frac{a_1^2}{I_1} + \frac{a_2^2}{I_2}}$$

- With that, we can write down the four after-collision speed components and the two after-collision roational velocities as

$$v_{1xe} = v_{1x} - \frac{F_x}{m_1} \qquad v_{2xe} = v_{1x} + \frac{F_x}{m_2}$$

$$v_{1ye} = v_{1y} \qquad v_{2ye} = v_{2y}$$

$$\omega_{1e} = \omega_1 + \frac{a_1 F_x}{I_1} \qquad \omega_{2e} = \omega_2 - \frac{a_2 F_x}{I_2}$$

- Another beautiful illustration of the first law of applied science! We have all the infinitely many outcomes of a tricky situation in just a few lines of equations. And we learned about the damage that can be produced (or avoided) by hitting objects with something like a sword, axe, golf club, baseball bat, ...

From here on it would be easy to move on to more complex collisions (3-dim; rough surfaces with frictions, ...) or into special "hitting" cases. Things like the percussion length would come into play but we have already looked at that in a separate module.
We could also, for example, figure out what kind of energy transfer coeffcients we have for moving *and* rotating objects. Let me clarify that a bit more. You can go right ahead - I will stay back, however.

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[1] Gros, Hauger, Schröder and Wall: Technische Mechanik 3. Springer, 9th edition.

[2] Once the Shogun gave a reception to honor the best swordsmen in Japan. All the top samurai were in attendance along with Court nobles and beautiful geisha. A geisha approached the third highest ranked swordsmen and asked; "Sir, can you demonstrate your sword skills for me?" At once, the samurai drew his sword and cut a hovering fly in half. "Very impressive", said the geisha. When she saw the samurai who was the second highest ranked swordsmen in Japan she asked him the same question. He immediately drew his sword and with two quick strokes quartered a fly. "Most impressive", said the geisha. Then she spotted the samurai that was the highest ranked swordsmen in all of Japan. "Honorable Sir", she said "would you be so kind as to demonstrate your sword skill for me?". The samurai drew his sword and cut into the air in the direction of a nearby fly, but the fly buzzed away. "Oh, so sorry you missed", said the geisha. "But I didn't miss", said the samurai humbly "that male fly will no longer be able to have offspring."