

Bending of Beams - Full Theory

[Link to: Bending of Beams - Easy Approach](#)

Defining the Ingredients

Science

What we want to calculate is the **deflection curve** of a bend beam for arbitrary forces or force distributions acting on that beam and for all kinds of (sensible) boundary conditions.

What we don't want to do is to make the enterprise more complicated than it must be. That means we use a number of approximations and assumptions that in non-mathematical terms simply mean:

- We do not look at situations where the beam bends severely, coming close to deforming or fracturing.
- The beam is much longer than any dimension of its cross-section.
- The beam is made from a homogeneous material, i.e. we do not, for example, consider layered composites
- We look only at the one-dimensional problem; bending a beam extending in the x -direction "down" the z axis. Nothing happens in the y -direction.

We do not need to start from scratch but can use the essential insights gained in the "[Beam bending - easy approach](#)" module.

These insights, in a nutshell, were:

- Local torques T bend beams locally.
- The amount of local bending can be measured by the radius of curvature r for the bend section.
- The cross-sectional shape and size is important but can be expressed in a single number that is called is the **area moment of inertia** I_A . We simply take that as given.
- The local torque $T(x)$ acting on a beam section at the coordinate x , and the local radius of curvature $r(x)$ of this beam section, are still related by our old equation from [part 1](#):

$$r(x) = \frac{I_A \cdot Y}{T(x)}$$

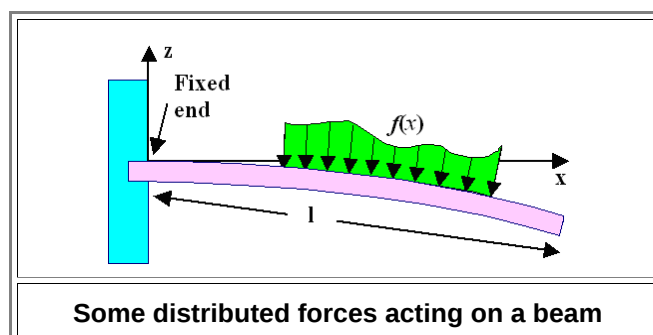
Y is Young's modulus of the material. What we need to do now is:

1. Calculate the local torques produced by the **exterior** forces
2. Relate the local radii of curvature that result from the local torques to the deflection curve (also called **elastic curve**) that we want to calculate.
3. Take care of all kinds of boundary conditions.

Let's start with the first point and consider forces, force distributions, and the torques resulting from those.

Forces, Force Distributions and Torques

In the [first figure](#) employed for this topic, a point force and a force distribution resulting from putting some brick on the beam, was shown. If you ponder that figure for a while, you will realize that we only need to consider force distributions in what follows because they contain point forces as a special case. To make the whole thing completely general, we allow arbitrary force distributions $f(x)$ as shown here:



- What can we say about the force distribution $f(x)$ in the drawing? You could produce it by putting down a brick extending from x_1 to x_2 that has a height profile as given by $f(x)$. We know that the **total** force it exerts is simply given by its weight (expressed in Newtons), and that implies that

$$F_{\text{to}} = \int_{x_1}^{x_2} f(x') \cdot dx' \quad \text{or} \quad f(x) = \frac{dF(x)}{dx}$$

- The left-hand equation with the integral is clear (I hope). The right-hand equation just reverses the situation. The force **distribution** $f(x)$ thus is simply the derivative of the total force you find as you go along the beam in x -direction.

How about the torque $T(x)$ that some force distribution $f(x)$ produces at the point x ?

- Tricky question. I won't go through the derivation but just give you the rather simple result:

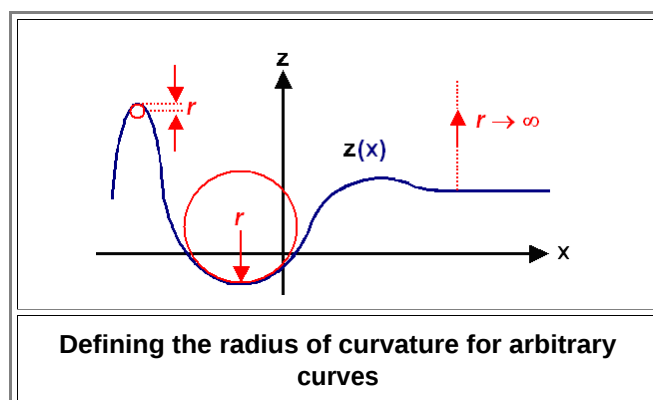
$$T(x) = \int_0^x F(x') \cdot dx' \quad \text{or} \quad \frac{dT(x)}{dx} = F(x) \quad \text{or} \quad \frac{d^2T(x)}{dx^2} = f(x)$$

We have what we need. We know what kind of torque is produced by external force distributions (and also point forces), and from our [old equation](#) we know how that relates to the local radius of curvature. The question now is how the local radius of curvature r relates to the deflection curve that we want to calculate.

Radius of Curvature and Differential Equation for the Deflection Curve

A rather fundamental insight from basic calculus is that **any** curve $y(x)$ can be assigned a local radius of curvature $r(x)$ at any point x .

- The figure below indicates schematically how one can "snugly" match some circle to any portion of the curve and thus assign a radius of curvature, including very small and large values, at any point.



- Since the local radius of curvature $r(x)$ is a well-defined property of the curve itself, the equation giving the curve should also allow to calculate $r(x)$. In other words: if you can draw it, you must be able to calculate it. As it turns out, you can calculate it - but the resulting equation, on first sight, is not all that simple. Calculus gives us

$$\frac{1}{r(x)} = \frac{z''}{(1 + z'^2)^{3/2}}$$

- with z' and z'' denoting, as usual, the first and second derivatives of the function $z(x)$

✓ Mercifully, it's not as bad as it looks. In regions where $y(x)$ is about parallel to the x -axis (as our beam line), a large radius of curvature *always* goes with a very small first derivative; see the figure above. Since we only bend gently, radii of curvature are going to be large, and we can simply assume that the first derivative of the deflection line is close enough to zero so we can ignore it.

This gives us 1) $1/r(x) \approx z''$ or, using our fundamental equation for the radius of curvature

$$\frac{1}{r(x)} = z''(x) = \frac{T(x)}{I_A(x) \cdot Y}$$

● Bingo! We have a second-order differential equation for $z(x)$, the deflection curve, that relates the second derivative of that function (that we are after, remember!) to some other functions of x .

I have even allowed the area moment of inertia I_A to be now a function of x , meaning that our beam may now change its cross-section along its length.

✓ We can even do better than this (or worse, depending on your point of view concerning differential equations). We know from right above that the torque $T(x)$ is just the second derivative of the force distribution $f(x)$. Rewriting slightly and differentiating inside our differential equation above twice more gives us

$$T''(x) = f(x) = \frac{d^2}{dx^2} (Y I_A(x) \cdot z''(x)) = Y I_A \cdot z''''(x) \quad \text{for } I_A = \text{const.}$$

● Bingo once more! The red part is the famous **beam bending equation**. It is a differential equation of *fourth* order, no less!

Mercifully, it is easy to solve - provided that the cross-section and thus I_A is constant and $f(x)$ is a simple function. If that is not the case things can get (mathematically) rather messy. A real beam has no problem whatsoever to find the solution. It just bends to the right shape.

✓ All that remains to do is to solve this equation for some interesting (and very simple) cases.

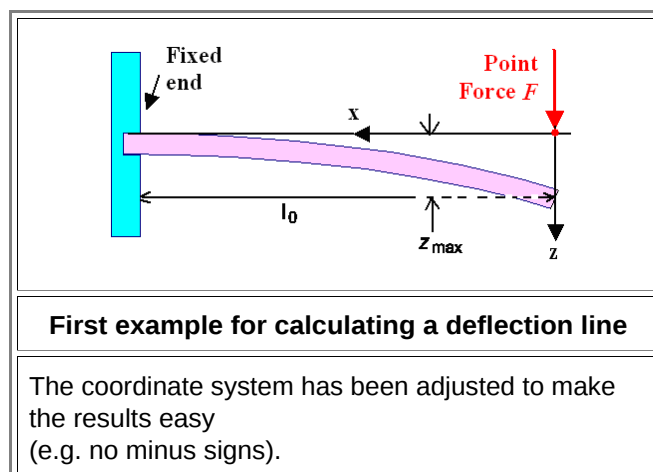
● If you know anything at all about differential equations, you know that for a fourth-order differential equation *four* integration constants will turn up that need to be determined from independent information.

The independent information that we need for *specific* solutions of a beam bending problem are (of course) the *boundary conditions*.

Let's look at a few examples to see how this works.

Bending Beams with Boundary Conditions

✓ Let's start by calculating the deflection line for a simple version of the the simple case used for illustrating the beam-bending problem. It is also of relevance for bending a sword.



Since we know for this simple case that the torque T is $F \cdot x$, we can start directly from the more simple [second-order differential equation](#). This is good because we (me included) do not know off-hand how to work with force distributions that are not distributed but "pointy" and thus would need delta-functions or other stuff from rather advanced math that we don't want to bother with here.

The decisive differential equation thus is

$$I_A Y \cdot z''(x) = F \cdot x$$

Everybody knows what to do now, of course. Just integrate twice, and you have

$$\begin{aligned} I_A Y \cdot z''(x) &= F \cdot x \\ I_A Y \cdot z'(x) &= \int F \cdot x \cdot dx = \frac{1}{2} F \cdot x^2 + C_1 \\ I_A Y \cdot z(x) &= \int (\frac{1}{2} F \cdot x^2 + C_1) dx = \frac{1}{6} F \cdot x^3 + C_1 \cdot x + C_2 \end{aligned}$$

C_1 and C_2 are the integration constants that we need to determine from the (*sensible!!*) boundary conditions. So what are the boundary conditions? Well, I already [gave those to you](#), but here we go again:

- $z(x = l_0) = 0$
- $dz(x = l_0) / dx = 0$

In words: There is no deflection at the place where the beam is fixed ($x = l_0$) for the coordinate system here and the slope of the bending line ($= dz / dx$) at the point is zero. Inserting that into the general solution above yields

$$\begin{aligned} I_A Y \cdot z'(l_0) = 0 &= \frac{1}{2} F \cdot l_0^2 + C_1 \Rightarrow C_1 = -\frac{1}{2} F l_0^2 \\ I_A Y \cdot z(l_0) = 0 &= \frac{1}{6} F \cdot l_0^3 - \frac{1}{2} F l_0^2 \cdot l_0 + C_2 \Rightarrow C_2 = \frac{1}{3} F l_0^3 \end{aligned}$$

Now you see what distinguishes "sensible" boundary conditions from not-so-sensible ones. We need two and only two boundary conditions to determine the integration constants. The real problem may not always give you that. Then you are stuck with a large and ugly can of worms that we simply won't open here.

Now let's celebrate the grand total for our first calculated bending line $z(x)$:

$$z(x) = \frac{1}{I_A Y} \left(\frac{1}{6} F \cdot x^3 - \frac{1}{2} F l_0^2 \cdot x + \frac{1}{3} F l_0^3 \right)$$

You are entitled to some exclamation now (Something between: "Oh my God" and "F.... off"; but stopping short of: "Let's kill him"). Remember what I said somewhere in [the beginning](#)? (No, it wasn't "let there be light", but thanks anyway).

Before I make things worse and go for the more complicated stuff, let's look at one major piece of enlightenment we can get from that result.

It is obvious from the geometry *and* from the equation that the **maximum deflection** is right at $x = 0$ or at the tip of your sword = beam. But only from the equation do we get the precise amount:

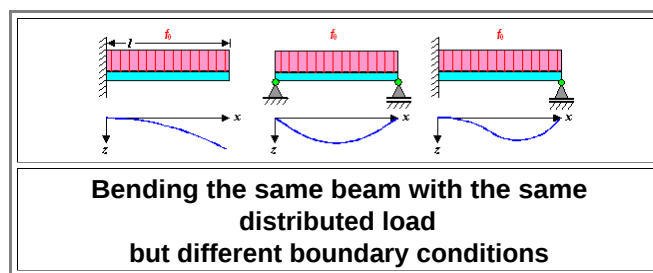
$$z_{\max}(x=0) = \frac{F \cdot l_0^3}{3 Y \cdot I_A} = \frac{4 F \cdot l_0^3}{Y \cdot b \cdot d^3} \quad \text{for a rectangular beam with width } b \text{ and thickness } d$$

Since [Young's modulus](#) Y is the same for all kinds of steel and iron, differences in how much you can "bend" a sword applying the same force depend *only* on:

- The third power of the length l_0^3 . Double the length and you deflect 8 times more.
- The areal moment of inertia. Look at this [table](#) and you see that making your sword half as thick (but twice as broad to keep the weight constant) will give you about 4 times more "flex".
- Changing the cross-section and thus the areal moment of inertia I_A as you go down the blade. This is not contained in the equation above but you can appreciate that it could have big effects.

Looking at Distributed Loads

For the grand finale let's look at some simple cases with a distributed load. A simple distributed load, being constant all over the beam for case as shown below.



Let's go straight to the differential equation and its solution:

$$Y I_A \cdot z''''(x) = f_0$$

$$Y I_A \cdot z'''(x) = F = f_0 \cdot x + C_1$$

$$Y I_A \cdot z''(x) = T = \frac{1}{2} f_0 \cdot x^2 + C_1 \cdot x + C_2$$

$$Y I_A \cdot z'(x) = \left(\frac{1}{6} \right) f_0 \cdot x^3 + \frac{1}{2} C_1 \cdot x^2 + C_2 \cdot x + C_3$$

$$Y I_A \cdot z(x) = \left(\frac{1}{24} \right) f_0 \cdot x^4 + \left(\frac{1}{6} \right) C_1 \cdot x^3 + \frac{1}{2} C_2 \cdot x^2 + C_3 \cdot x + C_4$$

Oh, the magic of differential equations! We have solved *all* problems involving a *constant* distributed load! But don't get excited about how easy it was! For some more complex function f_0 it would be far harder.

Anyway, all that remains to do now is to determine the four C 's, the integration constants. So let's look at the boundary conditions for the three cases given above.

First case

$z(0) = 0$	$C_4 = 0$
$z'(0) = 0$	$C_3 = 0$
$F(l_0) = 0$	$C_1 = -l_0 \cdot f_0$
$T(l_0) = 0$	$C_2 = \frac{1}{2} l_0^2 \cdot f_0$

- This need some explaining, The first two conditions are exactly the ones we [already considered](#) for the point load. They haven't changed. But we need two more boundary conditions to determine four integration constants. So we look at the boundary at right, the end of the beam. What is going on there? The answer is: nothing! There is *no* force or momentum, but hey, that's a condition, too. We obviously have $F(l) = T(l) = 0$. Both parameters correspond to integrating the distributed force once or twice, respectively, so we have no problem in using those "zero" conditions
- The elastic line $z(x)$ and the maximum deflection $z(l)$ now comes out to

$$z(x) = \frac{f_0 \cdot l^4}{24I_A Y} \left(\left(\frac{x}{l} \right)^4 - 4 \cdot \left(\frac{x}{l} \right)^3 + 6 \cdot \left(\frac{x}{l} \right)^2 \right)$$

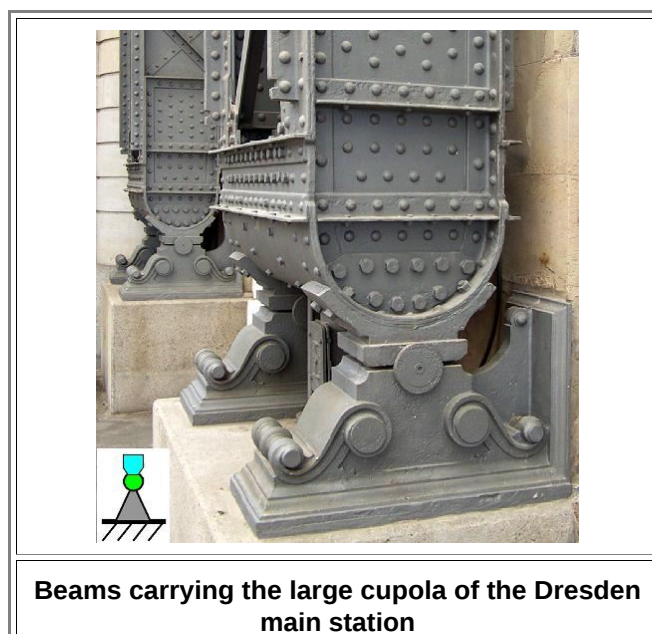
$$z(l) = \frac{f_0 \cdot l^4}{8I_A Y}$$

- If we compare that to the case of the point force, we just take $f_0 \cdot l = F$, the single force at the tip of the sword, and see that the ratio of maximum bending is **8/3**; deflection due to a single force is 2.66 times more pronounced than for a comparable distributed force.

Let's look at the two remaining cases. The boundary conditions and the C's determined from them are

Second case		Third case	
$T(0) = 0$	$C_2 = 0$	$z'(0) = 0$	$C_3 = 0$
$T(l) = 0$	$C_1 = -\frac{1}{2}l_0 \cdot f$	$z(0) = 0$	$C_4 = 0$
$z(0) = 0$	$C_4 = 0$	$T(l_0) = 0$	$\left(\begin{array}{l} C_1 = 5/8l_0 \cdot f \\ C_2 = 1/8l_0^2 \cdot f \end{array} \right)$
$z(l) = 0$	$C_3 = 1/24l_0 \cdot fl^3$	$z(l_0) = 0$	

- I refrain from writing down the final equations for the bending lines; the figure shows what they will look like quite nicely.
What those boundary conditions mean is graphically clear from the [figure](#), how they are expressed in equations from the table above. What the one on the left in the second case looks like in reality (for a perpendicular beam is shown in this picture:



Beams carrying the large cupola of the Dresden main station

The location is fixed but the torque will be zero since the beam can rotate "freely". If it couldn't, boundary conditions would no longer be "sensible" and large stresses could occur in the support.

What we have learned is that beam bending is not all that easy, even for the most simple problems and by using all kinds of approximations and tricks.

The bending of a real sword that has a cross-section that changes along its lengths, and for conditions where we bend a lot (e.g. the famous semi-circle around the smith) calculation would be difficult and only feasible numerically with a powerful computer.

So next time you see the steel skeleton of some skyscraper or any other mechanical constructions (like airplanes or cars) you realize that all you see are a lot of beams with different lengths and areal moments of inertia, hooked up in some way that defines the boundary conditions.

Rest assured that somebody has calculated the bending (and all the other deformations that might occur) of all those beams under the load they are going to bear, *and* has made sure that all the deflections stay within allowed limits. If they wouldn't, beams would be made stronger or more beams would be added. Of course, you can build a momentous pyramid or Gothic cathedral with doing any calculations of that sort. Just make sure that all load bearing parts are far larger than really needed. How large that is you find out by looking at pyramids or cathedrals that collapsed during their erection or shortly after (so they are no longer with us.) Use trial and error in other words. What you will *not* get in this way, however, are airplanes that actually fly because they would be too heavy.

Modern structural "beam-based" constructs are fully calculated, starting with the beam bending theory given here (and going far, far, beyond). That allows to make beams as light-weight as possible without compromising structural integrity. The fact that structural collapse essentially never happens anymore tells you that even super-complicated bending calculations can be done with precision. But only because nowadays we have the computer power needed to do that.

[Link to: Bending of Beams - Easy Approach](#)

¹⁾ I do know that there should be a minus sign but I omit it since it is not important. Other quantities that I used before also should have had a minus sign. In the end, choosing proper axes' orientations (z-axis downward, for example), everything comes without minus signs as written here