Exponentials and Logarithms

I hope you appreciate that this module is called "scientific" when in fact it should be called "basic" if not "elementary". Well, nobody is perfect, including me. I don't know a thing about organized sports or celebrities outside of science, for example. So I will explain exponentials and logarithms to you as you would explain the basics of baseball to me.

- I will move from extremely elementary math to heavier stuff. Challenge yourself a bit and see how far you get.

Science

Powers of Ten

If you need to deal with large numbers like the American deficit (around 14 trillion $ or 14.000.000.000.000 $), the numbers of time your kids asked "why" (3,5 gazillion), or the number of atoms in a sword (around 10.000.000.000.000.000.000.000.000), it gets tiresome to keep track of all the zeros. Not to mention that you are prone to make mistakes.
So mankind first invented **exponential notation** just as a simple way for abbreviating large numbers. Just write the number of zeros as "exponent" on the number ten by putting it up behind the **10** in a smaller font. $\mathbf{100 = 10^2}$, $\mathbf{100.000 = 10^5}$.
You just as well might interpret the **exponent** as the number of times you multiply the *base* **10** with itself. $\mathbf{10^3 = 10 \cdot 10 \cdot 10 = 1000}$. It's easy. The numbers above, for example, convert to:

- American deficit: around 14 trillion $ or $\mathbf{14 \cdot 10^{12} = 1,4 \cdot 10^{13}}$.
- Number of time your kids asked "why": **3,5** gazillion = $\mathbf{10^{\heartsuit}}$.
- Number of atoms in a sword: around $\mathbf{10^{25}}$.

So far, so easy. We have invented a simple way for abbreviating the writing of large numbers.

Surprise! It turns out that the exponential notation is far more powerful than you could have imagined in your wildest dreams. Let's look at the first and still fairly simple properties of exponential notations:

- **1.** It is also a great way to express *very small* numbers. A small number **SN** can always be written as **1** divided by a large number **LN**, **SN = 1/LN**. The **SN 0,0000001** can be written as $\mathbf{0,0000001 = 1/10.000.000 = 1/10^7 = 10^{-7}}$.
  That's right: write the exponential with a *minus* sign and now you count the zeros again but in the denominator of a fraction. Of course, you now loose the simple-minded interpretation of the meaning of the exponent, but so what.

- **2.** It is actually a great way to express *any number*. All we need to realize is that we have $\mathbf{1 = 10^0}$. Any number now can be written as $\mathbf{a \cdot 10^x}$ Let's look at examples:

  - $\mathbf{3,14 = 3,14 \cdot 10^0}$.
  - $\mathbf{31.400 = 3,14 \cdot 10^4}$.
  - $\mathbf{0,000314 = 3,14 \cdot 10^{-4}}$.

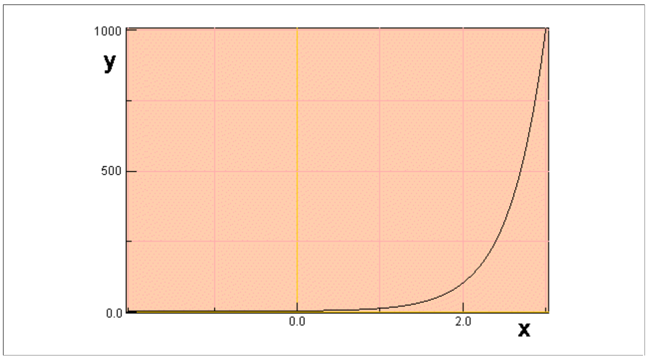
- **3.** Next, we discover that we have a *good* (but not yet *great*) way to *multiply* any two numbers like $\mathbf{a \cdot 10^x}$ times $\mathbf{b \cdot 10^y}$. The product is simply $\mathbf{ab \cdot 10^{x+y}}$.
  That *was* great! Tricky and time consuming multiplications have now been replaced by one easy multiplication and a simple addition.
  Nowadays computers do all that for you - and replacing a multiplication by an addition is actually the first step towards the way they really do it.
  Before I come to the *great* way for doing multiplications, I just mention that we also have covered division now. If you want to *divide* **A** by **B** you just as well could multiply **A** by **1/B**. What you get then is $\mathbf{a \cdot 10^x/(b \cdot 10^y) = (a/b) \cdot 10^{x-y}}$.

Now let's go on to a bit more tricky stuff. The motivation is that we do not like to multiply or divide the **a**'s and **b**'s. Well, *we* might not like it but *computers* can't even do it. All they can do is to add two numbers, provided they are 0 or 1.

- So let's enlarge the concept of exponentials and allow *any* number as exponent, not just *integers*.
  In other words: we consider expressions like $\mathbf{10^{1,5}}$, $\mathbf{10^{-1/4}}$, $\mathbf{10^{\pi}}$, and in particularly $\mathbf{10^{i}}$ with **i** being the unit of **imaginary numbers**, defined as $\mathbf{i^2 = -1}$.
  That's a bit mind boggling because the "counting the zeros" recipe now fails completely. The advantage of this slightly strange generalization is that we can now express *any* (positive) imaginable number **y** - large, small, integer, fraction, irrational, transcendent, imaginary, complex, whatever - by $\mathbf{y = 10^x}$ with **x** being a number from the list above.
  If we want a negative number we simply write $\mathbf{y = -10^x}$.

Hoppla, all of a sudden we have a *functional relationship* with x as variable: $\mathbf{y(x) = 10^x}$.

- No sweat. If we have some *known* ***x***, we can calculate the value of **y(*x*)**. For example we have $y(x = 2) = 10^2 = 100$, $y(x = -2) = 10^{-2} = 1/10^2 = 1/100 = 0{,}01$.
  Fine, but what about ***x* = 1,3**? Don't worry - be happy. There is a well-defined *algorithm* to compute **y** for *all* exponentials. Just look at the graph of the function $y(x) = 10^x$ below to find the answer, or trust your pocket calculator that will do it for you



The function $y = 10^x$ plotted out

- Now suppose you have a number like **3,14** and you would like to know what **x** would produce that number if taken as the exponent of **10**?
  In other words you have $y(x) = 10^x = 3{,}14$ and you want to solve that equation for ***x***.
  - Well, you can't. You need to invent **logarithms** first. Then the solution becomes **x = lg 3,74** and the "**lg**" stands for "logarithm to the base of 10".
    You probably don't know how to compute that? Don't worry, you probably don't know either how to compute **x = sin 3,74** or $x = (3{,}74)^{2{,}3}$. Nowadays you don't have to know how to do this kind of math because even the smallest and most stupid pocket calculator with some mathematical functions will know all of that. That's why we invented them in the first place. They just make life so much easier. Even if you *do* know how to do the jobs above, it is boring and tedious work.

Powers of Whatever

- If you paid attention, you noticed that in the example above I already sneaked in the next generalization. You do *not* need to take **10** as the *base* for putting an exponent to. You can take *any* number.
  - If your exponent is an integer, it is directly clear what to do. The exponent tells you how often to multiply the base with itself. $3{,}74^3 = 3{,}74 \cdot 3{,}74 \cdot 3{,}74 = \ldots$. If the exponent is *not* an integer, you just have the situation I discussed above - just a bit modified..
  - Now the mind should boggle a bit. We have discovered an infinity of possibilities to express *any* given number **y** by exponents: $y = x^a$ The base of our exponential description can be *any* number **x** you care to think of; all you need to do is find the proper exponent **a** that gives the **y** you are looking for.
- Infinitely many possibilities are not as cool as it sounds. Just a few close (girl) friends are better than infinitely many on facebook, for example. We must ask ourselves if some bases are better than others? Two obvious cases and one not-so-obvious case come to mind:

1. Base **x = 10**. That brings us back to where we started from. We like that base because we count in a system with 10 digits. Actually, we expressed numbers as sums of potencies of **10** all along - even if you may not have been aware of this. The number **1.234**, for example, simply means $1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0$, or $19 = 1 \cdot 10^1 + 9 \cdot 10^0$. For any number **y** you have $y = 10^x$ and **x = lg y** for the inverse relation.
   There is nothing particular special about the "10 system", except that you are going to invent it almost automatically if you are a counting species with 10 fingers.
2. If you are a species like computers that can only count to two (or computer scientists who can't count at all without hardware) you use only *two* digits for counting: 0 and 1. You have **2** as the base of your counting system. $19 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^1 + 1 \cdot 2^0$ or **1011**. We have $y = 2^x$ and **x = ld y** for the inverse relation; "***ld***" is the abbreviation for "**logarithm dualis**" or "logarithm to the base 2".
3. If you are a scientist or otherwise smart, you use the base **x = e**, sometimes known as **Euler's** number, with $e \approx 2{,}718281828\ldots$.
   We have $y = e^x$ and **x = ln y** for the inverse relation. The abbreviation "ln" stands for "**natural logarithm**", giving a hint that the base **e** is the most natural base to choose for doing exponentials.

- Why is **e** so prominent as a base for exponentials, and for about everything else in mathematical science? If you don't know this, I feel sorry for you. You have missed a lot in your life. I can't make up for this unfortunate turn of your biography in just a few lines. I thus will only give you a tiny taste treat.

- **1.** Pretty much all major equations describing the universe and what's going on on your planet are so-called differential equations. Typically, you want to determine an unknown function **y(x)**, but all you know is how that function and its derivatives relate to each other. For example, your differential equation might be $\mathbf{d^2y/dx^2 = A \cdot y}$.
  In words: the second derivative of the function **y** to be determined is equal to the function itself times some constant **A**. You can grasp that it would be helpful to know a function **y(x)** that is the same as its derivative, or **dy(x)/dx = y(x)**.
  There is one such function: $\mathbf{y(x) = e^x}$ (also written as **y(x) = exp(x)**)
  Now you know why "exponentials" figure extremely prominently in science. They come up more or less automatically as soon as you tackle problems; here is an example involving beer.
- **2.** A lot of what is going on in the world at large contains waves or oscillating things. If you activate this link, you can learn that in some way *everything* involves waves or oscillating things. Then you need to describe something like this  in equations.
  OK - you know that **y = sin x** will do the job. True enough - but trigonometric functions are difficult to deal with. That's why we turn them into exponentials, using one of the most remarkable equations of all of math:

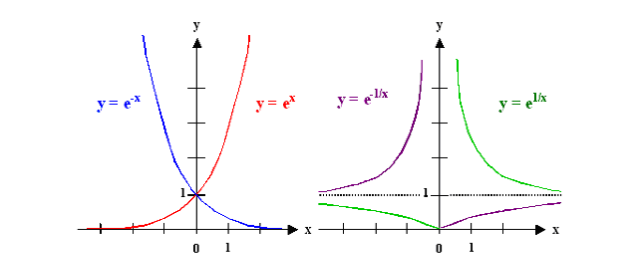
$e^{ix} = \cos x + i \cdot \sin x$	**Euler equation**
$e^{i\pi} + 1 = 0$	**Euler identity**

- The **Euler equation** tells you how to move from sinus and cosine functions to the far more simpler exponentials. The prize to pay is that you now must work with complex numbers **c** of the form **c = c' + ic''** with **i** = imaginary unit, typically but slightly wrongly called "square root of *minus* -1". Correct it is: $\mathbf{i^2 = -1}$. And no, it is not the same.
  This is not a real problem, however, because not only are calculations with complex numbers often far easier than with straight numbers, many problems (including pretty much all of quantum theory) are simply not solvable without complex numbers.
- The **Euler identity** results when you take **x** = π. It is a remarkable relation because it relates the five most important numbers in math, from which all others can be derived: One (**1**) and Zero (**0**), the base for all natural numbers; **i**, the base for imaginary numbers; and the *irrational* and *transcend* numbers π and **e** that "somehow" encode the deeper working of the universe and thus come up all the time.

Graphs and Equations

Here are graphs of the most important exponential functions. Don't use them to get numerical values since they are a bit qualitative concerning details.

- If you want the **ln** curves, just exchange **y** and **x**.



Graphs of the 4 most important exponential functions

Here are a few of the more important equations

$$e^x = \frac{1}{e^{-x}} \qquad (e^x)^y = e^{x \cdot y} \qquad \ln (x \cdot y) = \ln x + \ln y$$

$$e^x \cdot e^y = e^{x+y} \qquad (e^x)^{1/y} = e^{x/y} \qquad \ln \frac{x}{y} = \ln x - \ln y$$

$$\frac{e^x}{e^y} = e^{x-y} \qquad e^{\ln x} = x \qquad \ln x^y = y \cdot \ln x$$

Here are the two most astonishing relations, proving that the word "**natural** " with respect to the number **e** and its logarithm is apt:

$$y(x) = e^x \qquad \frac{dy(x)}{dx} = e^x$$

$$y(x) = \ln x \qquad \frac{dy(x)}{dx} = \frac{1}{x}$$

If you wonder how one actually calculates numbers, here are the relations. If you want to calculate the numerical value of **e**, just take **x = 1**.

- If this is your first encounter with infinite series, you have missed a lot of the wonders of math.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ..$$

$$\ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ..$$

$$\ln (1 - x) = - \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + .. \right)$$