Fourier Series and Transforms

Basics

Fourier Series

- Every (physically sensible) *periodic* function $\mathbf{f(t) = f(t+T)}$ with $\mathbf{T = 1/\nu = 2\pi/\omega}$ and ν, ω = frequency and angular frequency, respectively, may be written as a *Fourier series* as follows

$$f(t) \approx a_0/2 + a_1 \cdot \cos \omega t + a_2 \cdot \cos 2\omega t + \ldots + a_n \cdot \cos n\omega t + \ldots + b_1 \cdot \sin \omega t + b_2 \cdot \sin 2\omega t + \ldots + b_n \cdot \sin n\omega t + \ldots$$

  - and the Fourier coefficients a_k and b_k (with the index $k = 0, 1, 2, \ldots$) are determined by

$$a_k = 2/T \cdot \int_0^T f(t) \cdot \cos k\omega t \cdot dt$$

$$b_k = 2/T \cdot \int_0^T f(t) \cdot \sin k\omega t \cdot dt$$

- This can be written much more elegantly using complex numbers and functions as

$$f(t) = \sum_{-\infty}^{+\infty} c_n \cdot e^{in\omega t}$$

  - The coefficients c_n are obtained by

$$c_n = \int_0^T f(t) \cdot e^{-in\omega t} \cdot dt = \begin{cases} \tfrac{1}{2}(a_n - ib_n) & \text{for } n > 0 \\ \tfrac{1}{2}(a_{-n} + ib_{-n}) & \text{for } n < 0 \end{cases}$$

- The function $\mathbf{f(t)}$ is thus expressed as a sum of sin functions with the **harmonic frequencies** or simply *harmonics* $\mathbf{n \cdot \omega}$ derived from the fundamental frequency $\omega_0 = 2\pi/T$.
  - The coefficients c_n define the **spectrum** of the periodic function by giving the amplitudes of the harmonics that the function contains.

Fourier Transforms

- A nonperiodic function $\mathbf{f(t)}$ ("well-behaved"; we are not looking at some ***abominable*** functions only mathematicians can think of) can also be written as a Fourier series, but now the Fourier coefficients have some values for all frequencies ω, not just for some harmonic frequencies.
  - Instead of a spectrum with defined lines at the harmonic frequencies, we now obtain a **spectral density** function $\mathbf{g(\omega)}$, defined by the following equations

$$f(t) = \int_{-\infty}^{+\infty} g(\omega) \cdot e^{i\omega t} \cdot d\omega$$

$$g(\omega) = (1/2\pi) \cdot \int_{-\infty}^{+\infty} f(t) \cdot e^{-i\omega t} \cdot dt$$

The simplicity, symmetry and elegance (not to mention their usefulness) of these **Fourier integrals** is just amazing!