7.1 Introduction

An *ordinary differential equation* (for short: ODE) relates the derivative of a function that depends on a single real variable and that needs not be restricted to being single-valued (but may have vectors as results) to an expression that depends on the variable, contains the function itself and may also depend on other functions of the variable. In the most general form, a first-order ODE for a wanted function, denoted here as $\vec{y}$, can be written as

$$\frac{\mathrm{d}\vec{y}}{\mathrm{d}x} = f(x, \vec{y}) \tag{7.1}$$

where f is a known expression containing the dependency on the variable x and the wanted function itself, and it may contain other functions of the variable as well. For uniquely solving such a problem, also sufficient information about function values at a certain point (or at two different points) is needed, which altogether makes it an *initial value problem* (or a *two-point boundary value problem*). In the following, only initial value problems will be considered.

In general, any higher-order ODE can be transformed into a first-order ODE by introducing new functions for the lower-order derivatives, which effectively increases the vector size of the wanted function $\vec{y}$. In other words, when starting with an n-th order ODE of a single-valued function, this can be transformed into a system of n first order equations—which is an ODE for an n-component vector function $\vec{y}$.

Example:
The second-order differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y(x) = r(x) \tag{7.2}$$

can be transformed into a system of two first-order equations by introducing a vector function with the following two elements:

$$y_1 := y, \quad y_2 := \frac{\mathrm{d}y}{\mathrm{d}x}. \tag{7.3}$$

Obviously, for y_1 and y_2 it holds that

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}x} &= y_2(x), \\ \frac{\mathrm{d}y_2}{\mathrm{d}x} &= r(x) - p(x)y_2(x) - q(x)y_1(x). \end{aligned} \tag{7.4}$$