5.3 Partial Pivoting

For the Gauss elimination method to work it is necessary that at each step k there is a nonzero coefficient at variable x_k in line k, since otherwise one cannot use the k-th line to eliminate x_k in the remaining lines $i > k$. Moreover, since one must divide by the value of this coefficient $a_{kk}^{(k)}$ (the upper, bracketed index refers to the step number) to determine the multipliers m_{ik}, for a numerical computation it is important that this value is not too close to zero in order to prevent numerical errors becoming large.

Since $a_{kk}^{(k)}$ plays such a crucial role, it is called the pivot element. If it is zero or too small, one has to interchange the contents of the present row k with that of another row $j > k$ where the corresponding coefficient (at x_k) is neither zero nor too small. Numerically best suited for this is the row with the largest absolute coefficient, i.e. the largest of

$|a_{k+1\,k}^{(k)}|, |a_{k+2\,k}^{(k)}|, \ldots, |a_{nk}^{(k)}|$.

This choice of a new pivot element by a permutation of rows is called *partial pivoting*. Full pivoting would be to also interchange the columns of A in order to possibly obtain an even larger value for the new pivot element. As a result of partial pivoting, besides L and U one also obtains the row permutation matrix P (the result therefore being called an *LUP decomposition*); it holds that

$$\mathsf{PA} = \mathsf{LU}. \tag{5.11}$$

Example:

$$\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 3 & 8 & 13 \end{pmatrix} =: \mathsf{A}^{(1)} \;\rightarrow\; \mathsf{A}^{(2)} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 2 & 4 \end{pmatrix} \;\rightarrow\; \mathsf{PA}^{(2)} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} =: \mathsf{U} \tag{5.12}$$

The first-step multipliers are $m_{21} = 1$ and $m_{31} = 3$. To proceed, P has to interchange the 2nd and 3rd row, which also affects the construction of L. Since further elimination is not necessary, one has $l_{21} = 3$, $l_{31} = 1$, $l_{32} = 0$ ($l_{kk} = 1$). With these it can be verified that Eq. (5.11) holds.