

Figure 3.3: Newton method (two iterations). \bar{x} is the searched-for zero.

3.6 Newton Algorithm

Compared to the interval bisection method, this algorithm to find the zero \bar{x} of a function has the advantage that it shows much faster convergence (quadratic instead of linear). However, it has two severe drawbacks: Convergence is not always guaranteed, and one needs to determine the derivative of the function under investigation. Therefore, it is not often in practical use in its pure form. It is, however, very instructive to understand the basics of this algorithm, its convergence behavior, and why it may fail. That's why it is discussed here.

Consider a function f(x) being approximated by its tangent at x_0 . The zero x_1 of this tangent [which exists if $f'(x_0) \neq 0$] is an approximation for \bar{x} . To determine x_1 we express the slope of the tangent as

$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1} \Rightarrow x_0 - x_1 = \frac{f(x_0)}{f'(x_0)} \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$
(3.14)

This procedure is repeated with x_1 instead of x_0 to obtain an approximation x_2 , and so on; for a graphical representation see Fig. 3.3. Obviously, the iteration formula for the Newton method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \in \mathbb{N}_0, \quad f'(x_n) \neq 0.$$
 (3.15)

Compared to the formalism for fixed-point determination, this corresponds to the problem g(x) = x with g(x) = x - f(x)/f'(x). Therefore, according to the Banach fixed-point theorem, the Newton algorithm converges if $|g'(x)| \le q < 1$, or

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| \le q < 1,$$
(3.16)

holds in a certain interval containing the zero. Note that the Banach fixed-point theorem can yield linear convergence only. However, as will not be proved here, the Newton algorithm has quadratic convergence. The two main disadvantages of the Newton method are (i) the necessity to start the iteration rather close to the searched-for zero in order to achieve convergence and (ii) to calculate the inverse of the derivative at each iterative step. As long as one deals with functions of a single variable only, this is not severe, but for several variables the effort increases significantly.

The two main disadvantages of the Newton method are (i) the necessity to start the iteration rather close to the searched-for zero in order to achieve convergence and (ii) to calculate the inverse of the derivative at each iterative step. As long as one deals with functions of a single variable only, this is not severe, but for several variables the effort increases significantly.

On the other hand, the general aspect of using the derivative to determine the correct search direction is helpful in higher dimensions, and combined with a different method to determine the step size and to guarantee convergence (as, e.g., bisection) one obtains a method that works reliably and shows a super-linear convergence behavior. (If $|\bar{x} - x_{n+1}| \leq M |\bar{x} - x_n|^p$, p > 1, then the sequence $\{x_n\}$ is convergent of order p.)