2.3 Polynomial interpolation: Calculation according to Lagrange

We consider the following polynomials of degree n:

$$\begin{aligned}
\hat{L}_0(x) &= (x-x_1)(x-x_2)\cdots(x-x_n) \qquad (\hat{L}_0(x_0)\neq 0; \hat{L}_0(x_k)=0 \text{ for } k\neq 0),\\
\hat{L}_1(x) &= (x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n) \qquad (\hat{L}_1(x_1)\neq 0; \hat{L}_1(x_k)=0 \text{ for } k\neq 1),\\
&\vdots\\
\hat{L}_n(x) &= (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1}) \qquad (\hat{L}_n(x_n)\neq 0; \hat{L}_n(x_k)=0 \text{ for } k\neq n).
\end{aligned} \tag{2.1}$$

Short definition: $\hat{L}_k(x) = \prod_{j=0,\ j\neq k}^{n}(x-x_j)$. Next, we normalize the polynomials $\hat{L}_k(x)$ by division through $\hat{L}_k(x_k)$:

$$L_k(x) = \frac{\hat{L}_k(x)}{\hat{L}_k(x_k)}\,. \tag{2.2}$$

$L_k(x)$ is a polynomial of degree n, $L_k(x_j)=0$ for $j\neq k$, $L_k(x_k)=1$ (normalization). It can be written as

$$L_k(x) = \prod_{j=0,\ j\neq k}^{n} \frac{x-x_j}{x_k-x_j}\,. \tag{2.3}$$

Definition: *Lagrange polynomials*
The polynomials $L_k(x) = \prod_{j=0,\ j\neq k}^{n} \frac{x-x_j}{x_k-x_j}$ are called Lagrange polynomials.
Since $L_k(x_k)=1$ for all $k\in\{0,1,2,\ldots,n\}$, a polynom interpolating all data points $(x_0,y_0),(x_1,y_1),\ldots,(x_n,y_n)$ is obtained by

$$p(x) = y_0L_0(x) + y_1L_1(x) + y_2L_2(x) + \cdots + y_nL_n(x) = \sum_{j=0}^{n} y_jL_j(x)\,. \tag{2.4}$$

This can be easily seen by evaluating $p(x_k)$ for arbitrary k. The use of Lagrange polynomials is expedient if one has a fixed set of supporting points $x_0, x_1, \ldots, x_n$ and several sets of y data for these supporting points. Once the Lagrange polynomials are calculated, the desired function $p(x)$ can be found rather easily.

Example:
Consider again the following points: $(x_0,y_0)=(0,1)$; $(x_1,y_1)=(1,6)$; $(x_2,y_2)=(2,15)$. The corresponding Lagrange polynomials are:

$$\begin{aligned}
L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x-1)(x-2)\,,\\
L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-2)}{(1-0)(1-2)} = -x(x-2)\,,\\
L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{1}{2}x(x-1)\,.
\end{aligned} \tag{2.5}$$

With that, one obtains

$$\begin{aligned}
p(x) &= y_0L_0(x) + y_1L_1(x) + y_2L_2(x)\\
&= 1\times\tfrac{1}{2}(x-1)(x-2) + 6\times(-x)(x-2) + 15\times\tfrac{1}{2}x(x-1)\\
&= \tfrac{1}{2}(x^2-3x+2) - 6(x^2-2x) + \tfrac{15}{2}(x^2-x)\\
&= 1+3x+2x^2.
\end{aligned} \tag{2.6}$$