1.10 Numerical Errors: Basic definitions

Definition 1: *error* and *absolute error*
Consider a quantity x, and $\tilde{x}$ as an approximation for x, then $D_{\tilde{x}} = \tilde{x} - x$ is called the error of $\tilde{x}$ and $|D_{\tilde{x}}| = |\tilde{x} - x|$ the absolute error of $\tilde{x}$.

Examples:

$$\begin{array}{lrr} 1. & x = 2.1,\ \tilde{x} = 2 & \Rightarrow\ D_{\tilde{x}} = -0.1, |D_{\tilde{x}}| = 0.1 \\ 2. & x = \sqrt{2},\ \tilde{x} = 1.4 & \Rightarrow\ D_{\tilde{x}} = ? \\ 3. & x = \pi,\ \tilde{x} = 3.14 & \Rightarrow\ D_{\tilde{x}} = ? \end{array} \tag{1.12}$$

In general, the error cannot be determined exactly, because we don't know the exact value of x. In this case, we try a (possibly small) bound for specifying the absolute error.

Definition 2: *absolute maximum error*
Consider a quantity x, and $\tilde{x}$ as an approximation for x, and $a > 0$ a number with $|D_{\tilde{x}}| \leq a$, then a is an absolute maximum error of $\tilde{x}$.

Example:

$$\begin{array}{l} \text{Consider } x = \sqrt{2},\ \tilde{x} = 1.4;\ \text{estimation for } a? \\ \text{Since } 1.4^2 = 1.96 < 2 \text{ and } 1.42^2 = 2.0164 > 2, \\ \text{one has that } 1.4 < \sqrt{2} < 1.42,\ \text{therefore} \\ \left|\sqrt{2} - 1.4\right| = \sqrt{2} - 1.4 < 1.42 - 1.4 = 0.02 = a. \end{array} \tag{1.13}$$

(0.015 could also be derived as a maximum error for this example.)

Definition 3: *relative error*
Consider a quantity x, and $\tilde{x}$ as an approximation for x, $x \neq 0$, then $|D_{\tilde{x}}|/|x|$ is called the relative (percentage) error of $\tilde{x}$. Since x is often not known, the relative error is estimated by $|D_{\tilde{x}}|/|\tilde{x}|$.

Examples:

$$\begin{array}{lll} 1. & x & = 100\,\text{m} = 10000\,\text{cm},\ |D_{\tilde{x}}| = 1\,\text{cm}\ \Rightarrow\ \dfrac{|D_{\tilde{x}}|}{|x|} = \dfrac{1}{10000} = 0.0001 = 0.01 \\ 2. & x & = 2\,\text{cm},\ |D_{\tilde{x}}| = 1\,\text{cm}\ \Rightarrow\ \dfrac{|D_{\tilde{x}}|}{|x|} = \dfrac{1}{2} = 0.5 = 50 \end{array} \tag{1.14}$$

Definition 4: *significant digits*
The amount of significant digits is a measure of the *accuracy* of an approximative number. It is determined as follows:
1. Consider a quantity x, $|x| \geq 1$, and $\tilde{x}$ as an approximation for x, given as a decimal number with the digits $\pm a_k\, a_{k-1}\, \ldots\, a_1\, a_0 \,.\, a_{-1}\, a_{-2}\, \ldots\, a_{-n}$, i.e. each $a_i \in \{0, 1, ..., 9\}$ representing the value $a_i \times 10^i$, $a_k \neq 0$. Let j denote the smallest integer with $|\tilde{x} - x| \leq \frac{1}{2} \times 10^j$. Then we call the digit a_j of $\tilde{x}$ significant (or secure) and likewise, all the digits on the left of a_j.

Example:

$$\begin{array}{l} \text{Consider } x = 23.494321,\ \tilde{x} = 23.496;\ \text{how many significant digits?} \\ \text{Since } 0.0005 < |\tilde{x} - x| = 0.001679 \leq 0.005 = \frac{1}{2} \times 10^{-2},\ \text{one has that } j = -2. \\ \tilde{x} = 23.496\ \Rightarrow\ \text{start counting from the 9 until the beginning of the number} \\ \Rightarrow \text{4 significant digits (places).} \end{array} \tag{1.15}$$

2. Consider a quantity x, $|x| < 1$, and $\tilde{x}$ as an approximation for x, given as a decimal number with the digits $\pm 0.\, a_{-1}\, a_{-2}\, \ldots\, a_{-n}$, i.e. each $a_i \in \{0, 1, ..., 9\}$ representing the value $a_i \times 10^i$. Let j denote the smallest integer with $|\tilde{x} - x| \leq \frac{1}{2} \times 10^j$. Then we call the digit a_j of $\tilde{x}$ significant (or secure) and likewise, all the digits on the left of a_j up to the front digit $\neq 0$.

Example:

$$\begin{array}{l} \text{Consider } x = 0.02144,\ \tilde{x} = 0.02138;\ \text{how many significant digits?} \\ \text{Since } 0.00005 < |\tilde{x} - x| = 0.00006 \leq 0.0005 = \frac{1}{2} \times 10^{-3},\ \text{one has that } j = -3. \\ \tilde{x} = 0.02138\ \Rightarrow\ \text{start counting from the 1 to the left until the last digit} \neq 0 \\ \text{(i.e., until only zeros follow)} \Rightarrow \text{2 significant digits (places).} \end{array} \tag{1.16}$$

Remark: *accuracy* vs. *precision*
A number can be highly precise but completely inaccurate (many places, but none of them significant).