5.2.3 The Complex Index of Refraction

Dielectric Function and the Complex Index of Refraction

- Light is an electromagnetic wave. We have an electrical field that oscillates with some frequency (around 10^{15} **Hz** as you should now know by heart). If it impinges on a dielectric material (= no free electrons), it will jiggle the charges inside (bound electrons) around a bit. We looked at this in detail in **chapter 3**.
  - An electrical field caused some *polarization* of the dielectric material. This lead straight to the *dielectric constant* ϵ_r.

Attention!	The word "polarization" above and in chapter 3 has *nothing* to do with the "polarization" of light!	Attention!

  - Since the word "light" is synonymous to "oscillating electrical field", it is no surprise that ϵ_r is linked to the *index of refraction* $n = \epsilon_r^{1/2}$.
  - For oscillating electrical fields we needed to look at the *frequency dependence of the polarization* and that lead straight to the complex *dielectric function* $\epsilon_r(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$ instead of the simple dielectric constant ϵ_r. Go back to chapter 3.3.2 if you don't quite recall all of this.
- The dielectric function, after some getting used to, made life much easier and provided for new insights not easily obtainable otherwise. In particular, it encompassed the "ideal" dielectric losses *and* losses resulting from non-ideality. i.e. from a finite conductivity in its imaginary part.
- So it's logical to do exactly the same thing for the index of refraction. We replace n by a **complex index of refraction** n^* defined as

$$n^* = n + i\kappa$$

  - We don't use n' and n'' as symbols for the real and imaginary part but denote the real part by the (old) symbol n and the imaginary part by κ. This is simply to keep up with tradition and has no special meaning.
- We use the old relation between the index of refraction and the dielectric constant but now write it as

$$(n + i\kappa)^2 = \epsilon' + i\epsilon''$$

  - With $n = n(\omega)$; $\kappa = \kappa(\omega)$, since ϵ' and ϵ'' are frequency dependent as discussed before.
- Re-arranging for n and κ yields somewhat unwieldy equations:

$$n^2 = \frac{1}{2}\left(\left(\epsilon'^2 + \epsilon''^2\right)^{1/2} + \epsilon'\right)$$

$$\kappa^2 = \frac{1}{2}\left(\left(\epsilon'^2 + \epsilon''^2\right)^{1/2} - \epsilon'\right)$$

- Anyway - That is all. Together with the Fresnel equations we now have a lot of optics covered. Example of a real complex indexes of refraction are shown in the link.
  - So lets see how this works, and what κ, the so far unspecified imaginary part of n^*, will give us.

The Meaning of the Imaginary Part κ

- First, lets get some easier formula. In order to do this, we remember that ϵ'' was connected to the "dielectric" and static (ohmic) conductivity of the material and express ϵ'' in terms of the (total) conductivity σ_{DK} as

$$\epsilon'' = \frac{\sigma_{DK}}{\epsilon_0 \cdot \omega}$$

  - Note that in contrast to the definition of ϵ'' given before in the context of the dielectric function, we have an ϵ_0 in the ϵ'' part. We had, for the sake of simplicity, made a convention that the ϵ in the dielectric function contain the ϵ_0, but here it is more convenient to write it out, because then $\epsilon' = \epsilon_0 \cdot \epsilon_r$ is reduced to ϵ_r and directly relates to the "simple" index of refraction ***n***
  - Using that in the expression $(n + i\kappa)^2$ gives

$$(n + i\kappa)^2 = n^2 - \kappa^2 + i \cdot 2n\kappa = \epsilon' + i \cdot \frac{\sigma_{DK}}{\epsilon_0 \cdot \omega}$$

  - We have a complex number on both sides of the equality sign, and this demands that the real and imaginary parts must be the same on both sides, i.e.

$$n^2 - \kappa^2 = \epsilon'$$

$$n\kappa = \frac{\sigma_{DK}}{2\epsilon_0\omega}$$

  - Separating ***n*** and κ finally gives

$$n^2 = \frac{1}{2}\left(\epsilon' + \left(\epsilon'^2 + \frac{\sigma_{DK}^2}{4\epsilon_0^2\omega^2}\right)^{1/2}\right)$$

$$\kappa^2 = \frac{1}{2}\left(-\epsilon' + \left(\epsilon'^2 + \frac{\sigma_{DK}^2}{4\epsilon_0^2\omega^2}\right)^{1/2}\right)$$

  - Similar to what we had above, but now with basic quantities like the "relative dielectric constant" since $\epsilon' = \epsilon_r$ and the total conductivity σ_{DK}.
- Now lets look at the *physical* meaning of **n** and κ, i.e. the real and complex part of the complex index of refraction, by looking at an electromagnetic wave traveling through a medium with such an index.
  - For that we simply use the general formula for the electrical field strength ***E*** of an electromagnetic wave traveling in a medium with refractive index **n***. For simplicities sake, we do it one-dimensional in the ***x***-direction (and use the index "**x**" only in the first equation). In the most general terms we have

$$E_x = E_{0,x} \cdot \exp i \cdot (k_x \cdot x - \omega \cdot t)$$

  - With k_x = component of the wave vector in **x**-direction = $k = 2\pi/\lambda$, ω = circular frequency = $2\pi\nu$.
- There is no index of refraction in the formulas but you know (I hope) what to do.
  - You must introduce the velocity **v** of the electromagnetic wave in the material and use the relation between frequency, wavelength, and velocity to get rid of ***k*** or λ, respectively. In other words, we use

$$v = \frac{c}{n^*} \qquad v = \nu \cdot \lambda$$

$$\boldsymbol{k} = \frac{2\pi}{\lambda} = \frac{\omega \cdot n^*}{c}$$

- Of course, **c** is the speed of light in vacuum. Insertion yields

$$E_x = E_{0,x} \cdot \exp i \cdot \left(\frac{\omega \cdot n^*}{c} \cdot x - \omega \cdot t \right) = E_{0,x} \cdot \exp i \cdot \left(\frac{\omega \cdot (n + i \cdot \kappa)}{c} \cdot x - \omega \cdot t \right)$$

$$E_x = E_{0,x} \cdot \exp \cdot \left(\frac{i \cdot \omega \cdot n \cdot x}{c} - \frac{\omega \cdot \kappa \cdot x}{c} - i \cdot \omega \cdot t \right)$$

The red expression is nothing but the wavevector, so we get a rather simple result:

$$E_x = \underbrace{\exp - \frac{\omega \cdot \kappa \cdot x}{c}}_{\text{Decreasing amplitude}} \cdot \underbrace{\exp[i \cdot (k_x \cdot x - \omega \cdot t)]}_{\text{Plane wave}}$$

Spelt out: if we use a complex index of refraction, the propagation of electromagnetic waves in a material is whatever it would be for an ideal material with only a *real* index of refraction *times* a *attenuation factor* that decreases the amplitude exponentially as a function of depth ***x***.

- Obviously, at a depth often called **absorption length** or **penetration depth W = c/**ω **·** κ, the intensity decreased by a factor **1/e**.
- The imaginary part κ of the complex index of refraction thus describes rather directly the attenuation of electromagnetic waves in the material considered. It is known as **damping constant**, **attenuation index**, **extinction coefficient**, or (rather misleading) *absorption constant*. Misleading, because an absorption constant is usually the α in some exponential decay law of the form $I = I_0 \cdot \exp - \alpha \cdot x$ or what we called **W = c/**ω **·** κ above.
- Note: Words like "constant", "index", or "coefficient" are also misleading - because κ is not constant, but depends on the frequency just as much as the real and imaginary part of the dielectric function.

Using the Complex Index of Refraction

The equations above go beyond just describing the optical properties of (perfect) dielectrics because we can include all kinds of conduction mechanisms into σ, and all kinds of dielectric polarization mechanisms into **ε'**.

- We can even use these equations for things like the reflectivity of metals, as we shall see.

Keeping in mind that typical ***n***'s in the visible region are somewhere between **1.5 - 2.5** (***n*** ≈ **2.5** for diamond is one of the highest values as your girl friend knows), we can draw a few quick conclusions: From the simple but coupled equations for ***n*** and κ follows:

- For $\sigma_{DK} = 0$ (and, as we would assume as a matter of course, $\epsilon_r > 0$ (but possibly **< 1**?)) we obtain immediately $n = (\epsilon_r)^{1/2}$ and **κ = 0** - the old-fashioned simple relation between just ϵ_r and ***n***. Remember that $\sigma_{DK} = 0$ applies *only* if

  1. the static conductivity σ_{stat} is close to zero, and
  2. we have frequencies where **ε''** ≈ **0**, i.e well outside the resonance "peak" for optical frequencies.

Generally, we would *like* κ to be rather small for "common" optical materials!

- We also *expect* κ to be rather small for "common" optical materials, because optical materials are commonly insulators, i.e. so at least $\sigma_{static} \approx 0$ applies.

Let's look at some numbers now. With $\omega \approx 10^{16}$ **Hz** and $c = 3 \cdot 10^{10}$ **cm/s**, we have a penetration depth $W \approx 3 \cdot 10^{-6}/\kappa$.

- If, for example, the penetration depth should be in excess of **1 km** (for optical communication, say), $\kappa < 3 \cdot 10^{-11}$ is needed. It should be clear that this is quite a tough requirement on the material. How does it translates into requiremetns for σ_{DK} or ϵ''?

Exercise 5.2.3
Attenuation and dielectric function

If we now look at the other extreme, materials with large σ_{DK} values (e.g. metals), both ***n*** and κ will become large.

- Looking at the Fresnel equations we see that for large ***n*** values the intensity of the reflected beam approaches **100 %**, and large κ values mean that the little bit of light that is not reflected will not go very deep.
- Light that hits a good conductor thus will be mostly reflected and does not penetrate. Well, that is exactly what happens when light hits a metal, as we know from everyday experience.

Questionnaire
Multiple Choice questions to 5.2.3