4.3 Ferromagnetism

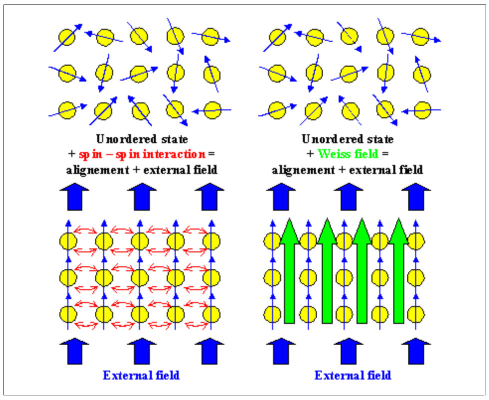
4.3.1 Mean Field Theory of Ferromagnetism

The Mean Field Approach

- In contrast to dia- and paramagnetism, *ferromagnetism* is of *prime importance* for electrical engineering. It is, however, one of the most difficult material properties to understand.
  - It is not unlike "ferro"electricity, in relying on strong interactions between neighbouring atoms having a permanent magnetic moment ***m*** stemming from the *spins* of electrons.
  - But while the interaction between electric dipoles can, at least in principle, be understood in classical and semi-classical ways, *the interaction between spins of electrons is an exclusively quantum mechanical effect with no classical analogon*. Moreover, a theoretical treatment of the three-dimensional case giving reliable results still eludes the theoretical physicists.
  - Here we must accept the fact that only **Fe**, **Co**, **Ni** (and some rare earth metals) show strong interactions between spins and thus ferromagnetism in elemental crystals.
  - In *compounds*, however, many more substances exist with spontaneous magnetization coming from the coupling of spins.
- There is, however, a relatively *simple theory of ferromagnetism*, that gives the proper relations, temperature dependences etc., - with one major drawback: It starts with an *unphysical assumption*.
  - This is the **mean field theory** or the **Weiss theory** of ferromagnetism. It is a phenomenological theory based on a central (wrong) assumption:

Substitute the elusive spin - spin interaction between electrons
by the interaction of the spins with a very strong magnetic field.

  - In other words, *pretend*, that in addition to your external field there is a *built-in magnetic field* which we will call the **Weiss field**. The Weiss field will tend to line up the magnetic moments - you are now treating ferromagnetism as an *extreme* case of paramagnetism. The scetch below illustrates this



- Of course, if the material you are looking at *is* a real ferromagnet, you don't have to *pretend* that there is a built-in magnetic field, because there *is* a large magnetic field, indeed. But this looks like mixing up cause and effect! What you want to result from a calculation is what you start the calculation with!
  - This is called a self-consistent approach. You may view it as a closed circle, where cause and effect loose their meaning to some extent, and where a calculation produces some results that are fed back to the beginning and repeated until some parameter doesn't change anymore.
  - Why are we doing this, considering that this approach is rather questionable? Well - it works! It gives the right relations, in particular the temperature dependence of the magnetization.
- The local magnetic field H_{loc} for an external field H_{ext} then will be

$$H_{loc} = H_{ext} + H_{Weiss}$$

  - Note that this has not much to do with the local electrical field in the Lorentz treatment. We call it "local" field, too, because it is supposed to contain everything that acts *locally*, including the modifications we ought to make to account for effects as in the case of electrical fields. But since our fictitious "Weiss field" is so much larger than everything coming from real fields, we simply can forget about that.

- Since we treat this *fictive* field H_{Weiss} as an internal field, we write it as a superposition of the external field H and a field stemming from the internal magnetic polarization J:

$$H_{loc} = H_{ext} + w \cdot J$$

  - With J = magnetic polarization and w = **Weiss's factor**; a constant that now *contains the physics of the problem*.

- This is the decisive step. We now identify the Weiss field with the magnetic polarization that is caused by it. And, yes, as stated above, we now do mix up cause and effect to some degree: the fictitiuos Weiss field causes the alignments of the individual magnetic moments which than produce a magnetic polarization that causes the local field that we identify with the Weiss field and so on.
  - But that, after all, *is* what happens: the (magnetic moments of the) spins interact causing a field that causes the interaction, that ....and so on . If your mind boggles a bit, that is as it should be. The magnetic polarization caused by spin-spin interactions and mediating spin-spin interaction just *is* - asking for cause and effect is a futile question.
  - The Weiss factor w now contains *all the local effects* lumped together - in analogy to the Lorentz treatment of local fields, μ_0, and the interaction between the spins that leads to ferromagnetism as a result of some fictive field.
  - But lets be very clear: *There is no internal magnetic field H_{Weiss} in the material* before the spins become aligned. This completely fictive field just leads - within limits - to the same interactions you would get from a proper quantum mechanical treatment. Its big advantage is that it makes calculations possible if you determine the parameter w experimentally.

- All we have to do now is to repeat the calculations done for paramagnetism, substituting H_{loc} wherever we had H. Lets see where this gets us.

Orientation Polarization Math with the Weiss Field

- The potential energy W of a magnetic moment (or dipole) **m** in an external magnetic field $\underline{H}$ now becomes

$$\begin{aligned} W &= - m \cdot \mu_0 \cdot (H + H_{Weiss}) \cdot \cos\varphi \\ &= - m \cdot \mu_0 \cdot (H + w \cdot J) \cdot \cos\varphi \end{aligned}$$

- The Boltzmann distribution of the energies now reads

$$N(W) = c \cdot \exp - \frac{W}{kT} = c \cdot \exp \frac{m \cdot \mu_0 \cdot (H + w \cdot J) \cdot \cos\varphi}{kT}$$

- The Magnetization becomes

$$\begin{aligned} M &= N \cdot m \cdot L(\beta) \\ &= N \cdot m \cdot L\left(\frac{m \cdot \mu_0 \cdot (H + w \cdot J)}{kT}\right) \end{aligned}$$

  - In the last equation the argument of **L(β)** is spelled out; it is quite significant that β contains $w \cdot J$.

- The total polarization is $J = \mu_0 \cdot M$, so we obtain the final equation

$$J = N \cdot m \cdot \mu_0 \cdot L(\beta) = N \cdot m \cdot \mu_0 \cdot L\left(\frac{m \cdot \mu_0 \cdot (H + w \cdot J)}{kT}\right)$$

- Written out in full splendor this is

$$J = N \cdot m \cdot \mu_0 \cdot \coth\left(\frac{m \cdot \mu_0 \cdot (H + w \cdot J)}{kT}\right) - \frac{N \cdot kT}{(H + w \cdot J)}$$

- What we really want is the magnetic polarization ***J*** as a function of the external field ***H***. Unfortunately we have a *transcendental* equation for ***J*** which can not be written down directly without a "***J***" on the right-hand side.
  - What we also like to have is the value of the spontaneous magnetization ***J*** for no external field, i.e. for **H = 0**. Again, there is no analytical solution for this case.
  - There is an easy graphical solution, however: We actually have *two* equations for which must hold *at the same time*:
  - The argument β of the Langevin function is

$$\beta = \frac{m \cdot \mu_0 \cdot (H + w \cdot J)}{kT}$$

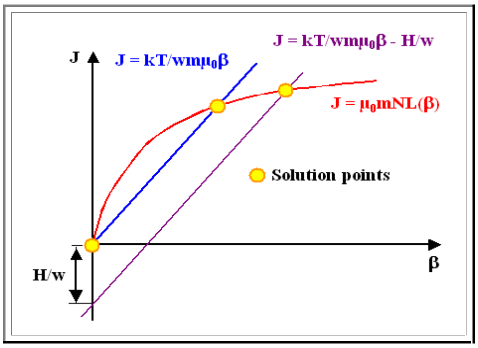
  - Rewritten for ***J***, we get our first equation:

$$J = \frac{kT \cdot \beta}{w \cdot m \cdot \mu_0} - \frac{H}{w}$$

  - This is simply a straight line with a slope and intercept value determined by the interesting variables ***H*, *w***, and ***T***.
- On the other hand we have the equation for ***J***, and this is our second independent equation
.

$$J = N \cdot m \cdot \mu_0 \cdot L(\beta) = N \cdot m \cdot \mu_0 \cdot L\left(\frac{m \cdot \mu_0 \cdot (H + w \cdot J)}{kT}\right)$$

  - This is simply the Langevin function which we know for any numerical value for β
- All we have to do is to draw *both* functions in a ***J* -** β diagram
  - We can do that by simply putting in some number for β and calculating the results. The intersection of the two curves gives the solutions of the equation for ***J***.
  - This looks like this



  - Without knowing anything about β, we can draw a definite conclusion:
  - For ***H* = 0** we have *two* solutions (or none at all, if the straight line is too steep): One for ***J* = 0** and one for a rather large ***J***.
  - It can be shown that the solution for ***J* = 0** is unstable (it disappears for an arbitrarily small field ***H***) so we are left with a *spontaneous large magnetic polarization* without an external magnetic field as the first big result of the mean field theory.
- We can do much more with the mean field theory, however.

- *First*, we note that switching on an *external magnetic field* does not have a large effect. J increases somewhat, but for realistic values of H/w the change remains small.
- *Second*, we can look at the *temperature dependence* of J by looking at the straight lines. For $T \to 0$, the intersection point moves all the way out to infinity. This means that all dipoles are now lined up in the field and **L(β)** becomes **1**. We obtain the *saturation value* J_{sat}

$$J_{sat} = N \cdot m \cdot \mu_0$$

- *Third*, we look at the effect of increasing *temperatures*. Raising T increases the slope of the straight line, and the two points of intersection move together. When the slope is equal to the slope of the Langevin function (which, as we know, is **1/3**), the two points of solution merge at $J = 0$; if we increase the slope for the straight line even more by increasing the temperature by an incremental amount, solutions do no longer exist and the spontaneous magnetization disappears.

This means, there is a *critical temperature* above which ferromagnetism disappears. This is, of course, the **Curie temperature** T_C.

- At the Curie temperature T_C, the slope of the straight line and the slope of the Langevin function for $\beta = 0$ must be identical. In formulas we obtain:

$$\frac{dJ}{d\beta} = \frac{kT_C}{w \cdot m \cdot \mu_0} = \text{slope of the straight line}$$

$$\left.\frac{dJ}{d\beta}\right|_{\beta = 0} = N \cdot m \cdot \mu_0 \cdot \frac{dL(\beta)}{d\beta} = \frac{N \cdot m \cdot \mu_0}{3}$$

- We made use of our old insight that the slope of the Langevin function for $\beta \to 0$ is **1/3**.

Equating both slopes yields for T_C

$$T_C = \frac{N \cdot m^2 \cdot \mu_0^2 \cdot w}{3k}$$

This is pretty cool. We did not solve an transcendental equation nor go into deep quantum physical calculations, but still could produce rather simple equations for prime material parameters like the Curie temerature.

- If we only would know w, the Weiss factor! Well, we do *not* know w, but now we can turn the equation around: If we know T_C, we can *calculate* the Weiss factor w and thus the *fictive magnetic field* that we need to keep the spins in line.
- In **Fe**, for example, we have $T_C = 1043$ K, $m = 2{,}2 \cdot m_{Bohr}$. It follows that

$$H_{Weiss} = w \cdot J = 1.7 \cdot 10^9 \text{ A/m}$$

- This is a truly *gigantic* field strength telling us that quantum mechanical spin interactions, if existent, are not to be laughed at.
- If you do not have a feeling of what this number means, consider the unit of H: A field of $1{,}7 \cdot 10^9$ **A/m** is produced if a current of $1{,}7 \cdot 10^9$ **A** flows through a loop (= coil) with **1** m^2 area. Even if you make the loop to cover only **1** cm^2, you still need $1{,}7 \cdot 10^5$ **A**.

We can go one step further and approximate the Langevin function again for temperatures $> T_C$, i.e. for $\beta < 1$ by

$$L(\beta) \approx \frac{\beta}{3}$$

- This yields

$$J(T > T_C) \approx \frac{N \cdot m^2 \cdot \mu_0{}^2}{3kT} \cdot (H + w \cdot J)$$

- From the equation for T_C we can extract w and insert it, arriving at

$$J(T > T_C) \approx \frac{N \cdot m^2 \cdot \mu_0{}^2}{3k(T - T_C)} \cdot H$$

- Dividing by H gives the susceptibility χ for $T > T_C$ and the final formula

$$\chi = \frac{J}{H} = \frac{N \cdot m^2 \cdot \mu_0{}^2}{3k \cdot (T - T_C)} = \frac{\text{const.}}{T - T_C}$$

- This is the famous Curie law for the paramagnetic regime at high temperatures which was a phenomenological thing so far. Now we derived it with a theory and will therefore call it **Curie - Weiss law**.

In summary, the mean field approach ain't that bad! It can be used for attacking many more problems of ferromagnetism, but you have to keep in mind that it is only a description, and not based on sound principles.

Questionaire
Multiple Choice questions to 4.3.1